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ORDER COMPARISONS ON CANONICAL ISOMORPHISMS

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Consider a nonnegative Hölder continuous 2-form $P(z)dxdy$ $(z = x + iy)$ on a connected Riemann surface R. We denote by $P(R)$ the linear space of solutions u of the equation $\Delta u = Pu$ on R and by $PX(R)$ the subspace of $P(R)$ consisting of those u with a certain boundedness property X. We also use the standard notations $H(R)$ and $HX(R)$ for $P(R)$ and $PX(R)$ with $P \equiv 0$. As for X we take B to mean the finiteness of the supremum norm $\|u\| = \sup_R |u|$, D the finiteness of the Dirichlet integral $D(u) = \int_R du \wedge {}^* du$, E the finiteness of the energy integral $E(u) = \int_R (du \wedge {}^* du + u^2(z)P(z)dxdy)$, and their nontrivial combinations BD and BE. Let $Q(z)dxdy$ be another 2-form of the same kind. We say that $PX(R)$ is canonically isomorphic to $QX(R)$ if there exists a linear isomorphism T of $PX(R)$ onto $QX(R)$ such that u and Tu have the same ideal boundary values for every u in $PX(R)$ in the sense that $|u - Tu|$ is dominated by a potential on R, i.e. a nonnegative superharmonic function whose greatest harmonic minorant is zero. In the pioneering work [14] concerning canonical isomorphisms, Royden proved the following *order comparison theorem*: If there exists a constant $c \geq 1$ such that

$$c^{-1}P(z) \leq Q(z) \leq cP(z) \tag{1}$$

on hyperbolic R except possibly for a compact subset K of R, then $PB(R)$ and $QB(R)$ are canonically isomorphic. In this connection we wish to discuss the following two questions:

1°. *Is the condition* (1) *also sufficient for* $PX(R)$ *and* $QX(R)$ *to be canonically isomorphic for* $X = D$, E, BD, *and* BE?

2°. *In the affirmative case how large can we make the exceptional set* K *in* (1) *for* $X = B$, D, E, BD, *and* BE?

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