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ON THE COHOMOLOGY OF CONGRUENCE SUBGROUPS OF SYMPLECTIC GROUPS

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§1. Introduction

This paper is concerned with the cohomology at "infinity" (in the sense of Harder [4], [5]) of a congruence subgroup of the symplectic group $G = Sp(2\ell, \boldsymbol{R})$. G is the subgroup of $GL(2\ell, \boldsymbol{R})$ consisting of matrices g satisfying ${}^t gJg = J$ where

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and I is the $\ell \times \ell$ identity matrix. We consider G as the real points of the algebraic group $\underline{G} = Sp(2\ell)$ defined over $\boldsymbol{Q}$. Let p be a prime not equal to 2 and Γ be the kernel of the natural map

$$Sp(2\ell, \boldsymbol{Z}) \longrightarrow Sp(2\ell, \boldsymbol{Z}/p^r\boldsymbol{Z}) .$$

We assume that r is chosen large enough so that Γ is torsion free.

We are interested in the Eilenberg-Maclane cohomology groups $H^*(\Gamma, \boldsymbol{C})$ of Γ (cf: Borel [2]). It is well-known that $H^*(\Gamma, \boldsymbol{C}) \approx H^*(X/\Gamma, \boldsymbol{C})$ where X is the symmetric space of maximal compact subgroups of G and G acts in a natural way on X. In [3] Borel and Serre constructed a compactification $\overline{X}/\Gamma$ of X/Γ having the property that $H^*(\overline{X}/\Gamma, \boldsymbol{C}) \approx H^*(X/\Gamma, \boldsymbol{C})$. $\overline{X}/\Gamma$ is a manifold with corners and is a union of subsets $e'(P)$ (see [3] p. 476) where P runs over the Γ conjugacy classes of parabolic $\boldsymbol{Q}$-subgroups of $\underline{G}$. Let

$$r\colon H^*(\overline{X}/\Gamma, \boldsymbol{C}) \longrightarrow H^*(\partial(\overline{X}/\Gamma), \boldsymbol{C}) \tag{1.1}$$

be the homomorphism induced by the map $\partial(\overline{X}/\Gamma) \to \overline{X}/\Gamma$. The general programme is to investigate the existence of a subspace $H^*_{\mathrm{inf}}(\Gamma, \boldsymbol{C})$ of $H^*(\Gamma, \boldsymbol{C}) \approx H^*(\overline{X}/\Gamma, \boldsymbol{C})$ which restricts isomorphically onto $\operatorname{Im} r$. The

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