K. Yamato
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A CHARACTERIZATION OF LOCALLY HOMOGENEOUS RIEMANN MANIFOLDS OF DIMENSION 3

KAZUO YAMATO

Introduction

It is classical to characterize locally homogeneous Riemann manifolds by infinitesimal conditions. For example, [Si] asserts that the local-homogeneity is equivalent to the existence of linear isometries between tangent spaces which preserve the curvatures and their covariant derivatives up to certain orders. It is also known that the local homogeneity is equivalent to the existence of a certain tensor field of type (1, 2) (for this and a further study, see [TV]).

In connection with his characterization theorem, Singer raised the following questions:

(Q1) What are the Riemann manifolds which are completely determined by their curvatures only?

(Q2) Do there exist curvature homogeneous spaces which are not locally homogeneous?

The purpose of the present paper is to give, in the 3-dimensional case, an explicit characterization (i.e. in terms of Riemannian invariants) of locally homogeneous Riemann manifolds, and to give some answers to the questions of Singer.

Our characterization is as follows: Let M be a connected, compact Riemann manifold of dimension 3 and S the Ricci tensor. Assume that the eigenvalues ρ_1, ρ_2, ρ_3 of S are constant on M (in other words we assume that M is curvature homogeneous).

THEOREM A. *Suppose that ρ_1, ρ_2, ρ_3 are distinct. Then M is locally homogeneous if and only if the 1-form $S \cdot \nabla S = \sum_{a,b} S^{ab} S_{ia;b}$ vanishes. If that is the case, then ρ_1, ρ_2, ρ_3 give complete isometry invariants for the universal covering manifold of M.*

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