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ON THE UNRAMIFIED EXTENSIONS OF THE PRIME CYCLOTOMIC NUMBER FIELD AND ITS QUADRATIC EXTENSIONS

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§ 1. Introduction

It is interesting to know what kinds of primes are the factors of the class number of an algebraic number field, and especially to find ones being prime to the degree. About this matter it is desirable to construct the unramified Abelian extensions plainly. In this paper we shall show some of them for the prime cyclotomic number field and its quadratic extensions using the units of subfields.

Let l be an odd prime and ζ be a primitive l-th root of unity. Let $k = \boldsymbol{Q}(\zeta)$ be the l-th cyclotomic number field over the field $\boldsymbol{Q}$ of rationals. If l is irregular, then there is an even integer r with $2 \leqq r \leqq l - 3$ such that the Bernoulli number B_{l-1-r} is divisible by l. In § 3 it will be proved that the existence of this even index r is equivalent to that of the cyclotomic unit in the subfield of k, of degree $(l-1)/(r, l-1)$, giving the unramified extension of k, of degree l by adjunction of its l-th root to k, under the assumption of Vandiver's conjecture on the second factor of the class number. When $l \equiv 1 \pmod 4$, this equivalence is related to N.C. Ankeny, E. Artin and S.D. Chowla's conjecture that $u \not\equiv 0 \pmod l$ for the fundamental unit $\varepsilon_l = (t + u\sqrt{l})/2 > 1$ of $\boldsymbol{Q}(\sqrt{l})$ which is not yet proved. We shall give in detail that $u \equiv 0 \pmod l$ if and only if $k(\sqrt[l]{\varepsilon_l})$ is unramified of degree l over k without Vandiver's conjecture.

In § 4 we shall consider a relative quadratic extension $K = k(\sqrt{d})$ where d is a square free rational integer prime to l. Let $l^* = (-1)^{(l-1)/2}l$. If d is a quadratic residue modulo l^2, then we shall give the elementary conditions to obtain the unramified Abelian extensions of degree l and l^2 over K by adjunctions of the l-th roots of the real units of $\boldsymbol{Q}(\sqrt{l^*}, \sqrt{d})$

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