T. Fujita
Nagoya Math. J.
Vol. 115 (1989), 105–123

REMARKS ON QUASI-POLARIZED VARIETIES

TAKAO FUJITA

Dedicated to Professor Morikawa on his 60th birthday

Introduction

Let V be a variety, which means, an irreducible reduced projective scheme over an algebraically closed field $\mathfrak{K}$ of any characteristic. A line bundle L on V is said to be *nef* if $LC \geqq 0$ for any curve C in V. Thus, "nef" is never an abbreviation of "numerically equivalent to an effective divisor". L is said to be *big* if $\kappa(L) = n = \dim V$. In case L is nef, it is big if and only if $L^n > 0$ (cf. [F7; (6.5)]. When L is nef and big, the pair (V, L) will be called a *quasi-polarized variety.*

We have $\chi(V, tL) = \sum_{j=0}^{n} \chi_j t^{[j]}/j!$ for some integers $\chi_0, \chi_1, \cdots, \chi_n$ where $t^{[j]} = t(t+1)\cdots(t+j-1)$ and $t^{[0]} = 1$. By Riemann-Roch Theorem we have $\chi_n = L^n$. Moreover, if V is normal, we have $-2\chi_{n-1} = (\omega + (n-1)L)L^{n-1}$ for a canonical divisor ω of V. We set $g(V, L) = 1 - \chi_{n-1}$, which is called the *sectional genus* of (V, L). We set $\Delta(V, L) = n + L^n - h^0(V, L)$, which is called the *$\Delta$-genus* of (V, L). We expect that we can describe the structure of (V, L) if Δ and/or g are small enough. When L is ample, we have the results in [F5], [F10], which we will generalize in this paper. Most results were announced in [F11].

In § 1 we show $\Delta \geqq 0$ for any quasi-polarized variety (V, L), and describe the structure of (V, L) with $\Delta = 0$ precisely. In particular $g = 0$ in this case. We conjecture the converse:

CONJECTURE. *$g \geqq 0$ for any quasi-polarized variety. Moreover, $g = 0$ implies $\Delta = 0$ if V is normal.*

This is completely unknown when $\mathrm{char}(\mathfrak{K}) > 0$, even if V is non-singular and L is ample. So, from § 2 on, we assume $\mathrm{char}(\mathfrak{K}) = 0$. In § 2 we give characterizations of $\boldsymbol{P}^n$ and hyperquadrics, which establish

Received July 22, 1988.