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NONCOMMUTATIVE CLASSICAL INVARIANT THEORY

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§1. Introduction

Let K be a field of characteristic zero, V a finite dimensional vector space and G a subgroup of $GL(V)$. The action of G on V is extended to the symmetric algebra on V over K,

$$K[V] = K \oplus V \oplus S^2(V) \oplus \cdots \oplus S^n(V) \oplus \cdots$$

and the tensor algebra on V over K,

$$K\langle V\rangle = K \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots .$$

Here $S^n(V)$ and $V^{\otimes n}$ denote the n-th symmetric power and n-th tensor power of V respectively.

We denote by $K[V]^G$ and $K\langle V\rangle^G$ the invariant ring of G acting on $K[V]$ and $K\langle V\rangle$, respectively. A main result of invariant theory says that, if G is linearly reductive, $K[V]^G$ is finitely generated. On the other hand Dicks and Formanek [2] proved that, if G is a finite group and not scalar, $K\langle V\rangle^G$ is not finitely generated. Lane [4] and Kharchenko [3] independently proved that, for arbitrary subgroup G of $GL(V)$, $K\langle V\rangle^G$ is a free associative K algebra.

In classical invariant theory one deals with the special linear group $SL(n)$. Consider the general n-ary form of degree r

$$f = \sum \frac{r!}{r_1! \cdots r_n!} a_{r_1\cdots r_n} x_1^{r_1} \cdots x_n^{r_n} , \qquad r_1 + \cdots + r_n = r ,$$

with coefficients $a_{r_1,\ldots,r_n}$ which are indeterminates over K.

If, for a linear transformation with determinant one, $x_1, \cdots, x_n$ undergo a linear transformation $x_i = \sum_j g_{ji} x'_j$, $g = (g_{ji}) \in SL(n)$, f is transformed into f of the form $f' = \sum r!/r_1! \cdots r_n!\, a'_{r_1\cdots r_n}, x_1'^{r_1} \cdots x_n'^{r_n}$. The mapping $a_{r_1\cdots r_n} \mapsto g(a_{r_1\cdots r_n}) = a'_{r_1\cdots r_n}$ defines a representation of $SL(n)$ on the vector space spanned by $a_{r_1\cdots r_n}$'s over K.

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