K. Enomoto
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UMBILICAL POINTS ON SURFACES IN R^N

KAZUYUKI ENOMOTO

Let $\psi: M \to R^N$ be an isometric imbedding of a compact, connected surface M into a Euclidean space R^N. ψ is said to be umbilical at a point p of M if all principal curvatures are equal for any normal direction. It is known that if the Euler characteristic of M is not zero and $N = 3$, then ψ is umbilical at some point on M. In this paper we study umbilical points of surfaces of higher codimension. In Theorem 1, we show that if M is homeomorphic to either a 2-sphere or a 2-dimensional projective space and if the normal connection of ψ is flat, then ψ is umbilical at some point on M. In Section 2, we consider a surface M whose Gaussian curvature is positive constant. If the surface is compact and $N = 3$, Liebmann's theorem says that it must be a round sphere. However, if $N \geq 4$, the surface is not rigid: For any isometric imbedding Φ of R^3 into R^4 $\Phi(S^2(r))$ is a compact surface of constant positive Gaussian curvature $1/r^2$. We use Theorem 1 to show that if the normal connection of ψ is flat and the length of the mean curvature vector of ψ is constant, then $\psi(M)$ is a round sphere in some $R^3 \subset R^N$. When $N = 4$, our conditions on ψ is satisfied if the mean curvature vector is parallel with respect to the normal connection. Our theorem fails if the surface is not compact, while the corresponding theorem holds locally for a surface with parallel mean curvature vector (See Remark (i) in Section 3).

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§ 1. Preliminaries

Let M be a connected n-dimensional C^∞ Riemannian manifold and let $\psi: M \to R^N$ be an isometric immersion of M into an N-dimensional Euclidean space R^N. Let D and $\bar{D}$ denote the covariant differentiations of M and R^N respectively. Let X, Y be tangent vector fields on M. Then

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