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SOLVABILITY OF THE DIOPHANTINE EQUATION $x^2 - Dy^2 = \pm 2$ AND NEW INVARIANTS FOR REAL QUADRATIC FIELDS

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In our recent papers [3, 4, 5], we defined some new D-invariants for any square-free positive integer D and considered their properties and interrelations among them. Especially, as an application of it, we discussed in [5] the characterization of real quadratic field $\mathbf{Q}(\sqrt{D})$ of so-called *Richaud-Degert* type in terms of these new D-invariants.

Main purpose of this paper is to investigate the Diophantine equation $x^2 - Dy^2 = \pm 2$ and to discuss characterization of the solvability in terms of these new D-invariants. Namely, we consider the equation $x^2 - Dy^2 = \pm 2$ and first provide necessary conditions for the solvability by using an additive property and the multiplicative structure of D (Proposition 2). Next, we provide necessary and sufficient conditions for the solvability in terms of an unit of the real quadratic field $\mathbf{Q}(\sqrt{D})$ (Theorems 1,2). Finally, we provide sufficient conditions for the solvability in terms of new D-invariants (Theorems 3,4). It is conjectured with a great expectation for these conditions to be also necessary conditions.

Throughout this paper, for any square-free positive integer D we denote by $\varepsilon_D = (t_D + u_D\sqrt{D})/2\ (>1)$ the fundamental unit of the real quadratic field $\mathbf{Q}(\sqrt{D})$ and by N the norm mapping from $\mathbf{Q}(\sqrt{D})$ to the rational number field $\mathbf{Q}$. Moreover, we denote $(\ /\ )$ the Legendre's symbol and by $[x]$ the greatest integer less than or equal to x.

On Pell's equation, we know already the following result by Perron (cf. [1], p. 106-109):

PROPOSITION 1 (O. Perron). *For any positive square-free integer $D \neq 2$, at most only one of the following three equations is solvable in integers:*

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