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OZEKI'S INEQUALITY AND NONCOMMUTATIVE COVARIANCE

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ABSTRACT. J.I.Fujii introduced the covariance of operators in Umegaki's theory of noncommutative probability. Very recently, it is observed that the so-called (noncommutative) covariance-variance inequality gives a unified method to prove certain operator inequalities including the celebrated Kantorovich inequality. Following after them, we shall discuss an operator version of Ozeki's inequality and consequently we show that the inequality needs a minor correction.

1. Introduction. From Umegaki's viewpoint [4] of noncommutative probability, M.Fujii, T.Furuta, R.Nakamoto and S.E.Takahashi [1] discussed the covariance and the variance of operators acting on a Hilbert space H. The covariance of two operators A and B (at a state $x \in H$) is defined by

$$\mathrm{Cov}(A,B) = (B^*Ax,x) - (Ax,x)(B^*x,x), \tag{1}$$

and the variance of A is defined by

$$\mathrm{Var}(A) = \|Ax\|^2 - |(Ax,x)|^2. \tag{2}$$

Their fundamental tool is the following covariance-variance inequality;

$$|\mathrm{Cov}(A,B)|^2 \leq \mathrm{Var}(A)\mathrm{Var}(B). \tag{3}$$

They observed that $\mathrm{Var}(A) \leq \frac{1}{4}(M-m)^2$ if A is a selfadjoint operator with $m \leq A \leq M$, and consequently they gave an estimation of the covariance by using (3): If $0 \leq m_1 \leq A \leq M_1$ and $0 \leq m_2 \leq B \leq M_2$, then

$$|\mathrm{Cov}(A,B)| \leq \frac{1}{4}(M_1-m_1)(M_2-m_2), \tag{4}$$

by which they unified proofs of many operator inequalities including the celebrated Kantorovich inequality.

Ozeki's inequality in [2] is the Kantorovich like inequality: Let a_i and b_i be two positive n-tuples, with $0 < m_1 \leq a_i \leq M_1$ and $0 < m_2 \leq b_i \leq M_2$ $(i = 1, \cdots, n)$ for some constants m_1, m_2, M_1, and M_2. Then the following inequality holds

$$(\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2) - (\sum_{k=1}^{n} a_k b_k)^2 \leq \frac{n^2}{4}(M_1M_2 - m_1m_2)^2. \tag{5}$$

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