On the Supports of Linearly Closed Convex Sets

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ABSTRACT. A condition for the supports of linearly closed convex sets to be closed is investigated.

1. Introduction

Let A be a convex subset of a real topological vector space. The frame of A is defined as the set

$$A_f = \{x \in A: \text{there exists } y \in A \text{ such that } y + t(x-y) \in A \text{ implies } t \leqq 1\},$$

and we denote $A^i = A \setminus A_f$. There exists a bounded closed convex set A with $A^i = \emptyset$. For example, in the space $C[0, 1]$ with supnorm, let A be the set of all points f in $C[0, 1]$ such that $f(0)=0$, $f(1)=1$ and $0 \leqq f(x) \leqq 1 (x \in [0, 1])$ then A is a bounded closed convex set with $A^i = \emptyset$. A convex set A is called to be linearly closed if for any two points x and $y (x \neq y)$ of A, the interesction of A and the line through x and y has two extreme points. A set consists simply of one point is also called to be linearly closed. Bounded closed convex subsets of a Hausdorff topological vector space are linearly closed, but the converse is not true. For example, in the space $C[0, 1]$ with supnorm, the set of all points f in $C[0, 1]$ such that $f(x)=0$ on a neighbourhood of 0, $f(x)=1$ on a neighbourhood of 1 and $0 \leqq f(x) \leqq 1 (x \in [0, 1])$ is not closed but linearly closed.

A support S of A is a nonempty convex subset of A which satisfies the condition that if an interior point of a line segment $[x, y]$ in A belongs to S, then $[x, y] \subset S$. A itself is a support of A. A support of A which is not equal to A is called a non-trivial support of A. No point of A^i is contained in the non-trivial support of A.

2. Statement of theorem

LEMMA 1. *The frame A_f of a convex set A has the following property: if an interior point of a line segment $[x, y]$ in A belongs to A_f, then $[x, y] \subset A_f$.*

PROOF. It is sufficient to prove that if $x \in A^i$, then for each y in A, $[x, y)$ is contained in A^i. Let x be an element of A^i. It is easy to see that for any two points z and w of A, there exists $s > 0$ such that $|t| < s$ implies that $x + t(z-w)$ belongs to A. For each y in A, let $w = \lambda x + (1-\lambda)y$ $(0 \leqq \lambda \leqq 1)$. Since