Nihonkai Math. J.
Vol.16(2005), 23-29

AN EXTENSION OF KANTOROVICH INEQUALITY

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Dedicated to the memory of Professor Shizuo Kakutani

ABSTRACT. A simple proof of the Kantorovich inequality is presented, and consequently an extension of the inequality is proposed which seems neat.

1. In this note an *operator* means a bounded linear operator acting on a Hilbert space. For a positive invertible operator A, the interval $I = [m, M]$ is the convex hull of the spectrum of A. Let f be a (real-valued) continuous function defined on I and μ a probability measure on I, then the expectation value is defined by $\mathrm{E}[f] = \int_I f(t)d\mu(t)$. For the convenience, by the spectral theorem, an operator A is identified with the function t, $f(A)$ with $f(t)$, and the scalars are identified with the scalar multiples of the identity operator.

In these circumstances, the celebrated Kantorovich inequality is written as follows:

$$(1) \qquad (Ax, x)(A^{-1}x, x) \le \frac{(M+m)^2}{4Mm}, \quad \text{for a unit vector } x \in H.$$

There are a lot of proofs of the inequality [10], [14], [16] - [18], etc. Among them, the proof in [14] presents the following equivalent inequality:

$$(2) \qquad \mathrm{E}[t]\mathrm{E}[1/t] \le \frac{(M+m)^2}{4Mm}.$$

Let us cite the proof of (2) in [14]. Put

$$l(t) = \frac{M+m-t}{Mm},$$

then $1/t \le l$, so that $\mathrm{E}[1/t] \le \mathrm{E}[l]$, and

$$\mathrm{E}[t]\mathrm{E}[1/t] \le \mathrm{E}[t]\mathrm{E}[l] = \mathrm{E}[t] \cdot l(\mathrm{E}[t]) = \frac{1}{Mm}\left((M+m)\mathrm{E}[t] - \mathrm{E}[t]^2\right).$$

Since the last term is a quadratic polynomial in $\mathrm{E}[t]$ and approaches its maximum at $\mathrm{E}[t] = (M+m)/2$, the desired (2) is proved.

Observing the above proof, we see that the essential tools are linearity and monotonicity of the expectation.

There are a large number of authors who have presented extensions of the Kantorovich inequality [2] - [6], [8] - [12], [14] - [18], etc..

In this note we shall modify the above proof in [14] to show an extension of Kantorovich inequality.

2000 *Mathematics Subject Classification.* 47A30, 47A63.
Key words and phrases. Kantorovich inequality, Kantorovich constant, Specht ratio.