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ON SOME OPERATORS WHOSE PRODUCTS ARE POSITIVE

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ABSTRACT. Let A, B be bounded linear operators on $\mathcal{H}$ satisfying

$$AB \geq 0,\ A^2B \geq 0,\ AB^2 \geq 0.$$

We study the positivity of A and B under the condition $\mathrm{Ker}AB = \{0\}$ and the representation for contractions A, B using positive operators.

It is known that a bounded linear operator T on a Hilbert space $\mathcal{H}$ which satisfies $T^n \geq 0$ $(n \geq 2)$ is not necessarily positive. In fact, if T satisfies that T^2, T^3 are positive, then T can be decomposed into a direct sum of operators N and S such that $N^2 = 0, S \geq 0$ (cf. [2]). So it is clear that $T^n \geq 0$ $(n \geq 2)$ and $\mathrm{Ker}T = \{0\}$ imply the positivity of T. This result motivates the following conjecture:

For bounded linear operators A and B on $\mathcal{H}$ satisfying

$$(*) \qquad AB \geq 0,\ A^2B \geq 0 \text{ and } AB^2 \geq 0,$$

if it holds $\mathrm{Ker}AB = \{0\}$, then both A and B are positive.

We can easily see that this conjecture fails without the assumption $\mathrm{Ker}AB = \{0\}$ (see Example). As stated in the following, in many cases the above conjecture is true. But, in general, we do not know whether the assumption $(*)$ and $\mathrm{Ker}AB = \{0\}$ imply the positivity of A and B or not. So our aim is, under these assumptions, to give a sufficient condition which implies their positivity.

Throughout this paper, we assume that bounded linear operators A and B satisfy the condition $(*)$.

Lemma 1. *If* $\overline{\mathrm{Ran}B} = \mathcal{H}$, *then* $A \geq 0$. *Similarly, if* $\overline{\mathrm{Ran}A^*} = \mathcal{H}$, *then* $B \geq 0$.

Proof. By the assumption, we get

$$AB^2 = (AB^2)^* = B^*(AB)^* = B^*AB.$$

So we have,

$$\langle ABx|Bx\rangle = \langle B^*ABx|x\rangle = \langle AB^2x|x\rangle \geq 0,$$

for all $x \in \mathcal{H}$. Thus the condition $\overline{\mathrm{Ran}B} = \mathcal{H}$ implies $A \geq 0$.

Since $AB=(AB)^*=B^*A^*$, if we consider A and B instead of B^* and A^*, then the condition $\overline{\mathrm{Ran}A^*} = \mathcal{H}$ implies $B \geq 0$. □

We remark that the assumption $(*)$ implies the positivity of A^nB and AB^n $(n = 1, 2, \dots)$ by the similar argument in the above proof.

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