Normal Structure of Weakly Compact Sets of Banach Spaces

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Abstract. We obtain the geometrical conditions for weakly compact sets to be normal.

1. Introduction

A subset C of a Banach space E is said to be normal if there exists x in C such that

$$\sup\{\|x-y\|: y \in C\} < \operatorname{diam} C,$$

and a convex set $D \subset E$ is said to have normal structure if every nontrivial convex subset of D is normal. A Banach space E is said to have weak normal structure if every weakly compact convex subset of it has the normal structure.

The concept "normal structure" has relations with existence of fixed points of nonexpansive maps. Browder [1] proved that a nonexpansive self-map on a bounded closed convex subset of a uniformly convex Banach space has a fixed point. Bounded closed convex subsets of a uniformly convex Banach space are weakly compact, and uniformly convex Banach spaces have the normal structure. It is known that if a Banach space E has the weak normal structure, then every nonexpansive self-map on a weakly compact convex subset of E has a fixed point ([4]).

A Banach space is said to be uniformly convex (UC) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x\| \leqq 1,\ \|y\| \leqq 1,\ \|x-y\| > \varepsilon \Rightarrow (\|x+y\|)/2 < 1-\delta.$$

$B_r(x)$ denotes the closed ball of radius r with center x.

Sufficient conditions of a Banach space to have the weak nermal structure were obtained successively as follows:

(1) nearly uniformly convex (NUC): for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x_n\| \leqq 1,\ \inf\{\|x_n - x_m\| : n \neq m\} \geqq \varepsilon \Rightarrow \operatorname{co}\{x_n\} \cap B_{1-\delta}(0) \neq \phi,$$

where $\operatorname{co}\{x_n\}$ denotes the covex hull of $\{x_1, x_2, \dots\}$ ([3]),

(2) uniformly Kadec-Klee (UKK): for each $\varepsilon > 0$ there exists $\delta > 0$ such that