Nihonkai Math. J.
Vol.9 (1998), 111–115

METRICS ON S^3 SUCH THAT BRIESKORN CURVE IS A GEODESIC

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1. Introduction.

Let $(a_1, a_2, \dots, a_{n+1})$ be an $(n+1)$-tuple of positive integers with $a_i \geq 2\,(i = 1, 2, \dots, n+1)$. $B^{2n-1}(a_1, a_2, \dots, a_{n+1}) \subset \mathbb{C}^{n+1}$ is said to be a $(2n-1)$-dimensional Brieskorn manifold if it satisfies the following two equations.

$$|z_1|^2 + |z_2|^2 + \cdots + |z_{n+1}|^2 = 1, \tag{1}$$

$$(z_1)^{a_1} + (z_2)^{a_2} + \cdots + (z_{n+1})^{a_{n+1}} = 0. \tag{2}$$

In particular, $B^1(a_1, a_2)$ is called a Brieskorn curve on a unit sphere S^3.

Brieskorn manifolds have interesting properties in the topological and differential view points. For example they have S^1-actions with the singular orbits as the G-manifolds, some of these are exotic spheres and they have many normal contact metric structures(cf.[2],[4] and [5]).

Let x^1, x^2, x^3, x^4 be a local coordinate system of $\mathbb{R}^4$. We put $x^1 = \cos\theta^1$, $x^2 = \sin\theta^1 \cos\theta^2$, $x^3 = \sin\theta^1 \sin\theta^2 \cos\theta^3$, $x^4 = \sin\theta^1 \sin\theta^2 \sin\theta^3$, where $\theta^1, \theta^2 \in (0, \pi)$ and $\theta^3 \in (-\pi, \pi)$. Then the usual metric on S^3 is defined by

$$ds^2 = (d\theta^1)^2 + \sin^2\theta^1 (d\theta^2)^2 + \sin^2\theta^1 \sin^2\theta^2 (d\theta^3)^2. \tag{3}$$

In general, Brieskorn curve $B^1(p, q)$ is not a geodesic on S^3 with the usual metric (3).

The purpose of the present paper is to describe an adapted metric g such that Brieskorn curve $B^1(p, q)$ is a geodesic on sphere S^3.

Theorem. *A metric on S^3 such that Brieskorn curve $B^1(p, q)$ is a geodesic is given by*

$$ds^2 = p^2 (d\theta^1)^2 + p^2 \sin^2\theta^2 (d\theta^2)^2 + q^2 \sin^2\theta^1 \sin^2\theta^2 (d\theta^3)^2,$$

where p and q are integers with $p \geq q \geq 2$.

Moreover Brieskorn curve $B^1(p, q)$ is

$$(\exp(\sqrt{-1}s/p),\ \exp(\sqrt{-1}s/q))B_0 \quad \textit{for all } s,$$

where B_0 is a point of $B^1(p, q)$.

The auther express his sincere thanks to Professors Y. Hatakeyama and T. Takahashi for his valuable advice, guidance and encouragement.