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AN OPERATOR VERSION OF THE WILF-DIAZ-METCALF INEQUALITY

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ABSTRACT. Diaz and Metcalf generalized the Wilf inequality, which is also a generalization of the arithmetic-geometric mean inequality, to the case of vectors in a Hilbert space. In this note, we shall consider Wilf-Diaz-Metcalf type inequalities for operators on a Hilbert space.

1. Introduction. In 1963, Wilf [11] generalized the arithmetic-geometric mean inequality for complex numbers and Diaz and Metcalf [5] advanced it to the case for vectors in Hilbert space by the similar proof to Wilf's one:

Theorem A. *Let a be a unit vector in a Hilbert space H. If nonzero vectors x_k in H satisfy*

$$0 \le r \le \frac{\mathrm{Re}\,(x_k, a)}{\|x_k\|}$$

for some r, then

$$r(\|x_1\| \cdots \|x_n\|)^{1/n} \le \frac{\|x_1 + \cdots + x_n\|}{n}.$$

More presicely, they showed the following inequality,

$$r(\|x_1\| + \cdots + \|x_n\|) \le \|x_1 + \cdots + x_n\|,$$

which implies Theorem A by the arithmetic-geometric mean inequality.

In this note, we try to generalize the above inequalities to the case for operators on a Hilbert space on a line with their proof.

2. The Wilf-Diaz-Metcalf inequality. An operator version of Theorem A would be the following one:

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