Nihonkai Math. J.
Vol.8 (1997), 91-94

A free group of rotations with rational entries on the 3-dimensional unit sphere

SATÔ Kenzi

ABSTRACT. Dekker showed the existence of a free group of rank 2 of rotations (special orthogonal matrices) acting on the 3-dimensional unit sphere whose non-trivial rotations does not have a fixed point. In this paper, we find a free group on the 3-dimensional unit sphere which satisfies the same condition and whose entries of matrices are rational.

Introduction.

Banach and Tarski proved a very curious paradox which enables us to partition a golf ball into finite number of pieces and reconstruct the earth from these pieces:

Paradox A [BT,W]. *If U and V are any bounded subsets of the 3-dimensional Euclidean space $\mathbb{R}^3$, each having non-empty interior, then U and V can be both partitioned into the same finite number of respectively (orientation-preserving) congruent pieces. Formally,*

$$U = \bigcup_{l=0}^{m-1} U_l, \quad V = \bigcup_{l=0}^{m-1} V_l,$$

$U_l \cap U_{l'} = \emptyset = V_l \cap V_{l'}$ *if* $0 \leqq l \lneqq l' \lneqq m$, *and there are (orientation-preserving) isometries* $\alpha_0, \ldots, \alpha_{m-1}$ *such that, for each* $0 \leqq l \lneqq m$, $\alpha_l(U_l) = V_l$.

Paradox A owes the following paradox essentially:

Paradox B [Si,W]. *The 2-dimensional unit sphere* $\mathbb{S}^2 = \{\vec{r} \in \mathbb{R}^3 : |\vec{r}| = 1\}$ *admits three decompositions into disjoint pieces*

$$\mathbb{S}^2 = A_0 \cup \cdots \cup A_{p-1} \cup B_0 \cup \cdots \cup B_{q-1},$$

$$\mathbb{S}^2 = A'_0 \cup \cdots \cup A'_{p-1}, \quad \text{and} \quad \mathbb{S}^2 = B'_0 \cup \cdots \cup B'_{q-1},$$

and $A_i \approx A'_i$ $(i = 0, 1, \ldots, p-1)$, $B_j \approx B'_j$ $(j = 0, 1, \ldots, q-1)$, *where* $C \approx C'$ *means* $\gamma(C) = C'$ *for a suitable rotation* γ.

Robinson [R,W] constructed decompositions for $p = q = 2$, which is the fewest piece's decompositions. For a given set E, it is a very important problem to find

1991 *Mathematics Subject Classification*. Primary 20E05, 20H05, 20H20; Secondary 15A18, 51F20, 51F25.