Nihonkai Math. J.
Vol.8 (1997), 117-122

OPERATOR INEQUALITIES RELATED TO CAUCHY-SCHWARZ AND HÖLDER-McCARTHY INEQUALITIES

Masatoshi Fujii*, Saichi Izumino, Ritsuo Nakamoto*** and Yuki Seo******

Abstract. We give an improvement of the Cauchy-Schwarz inequality, which is based on the covariance-variance inequality. We also give a complementary inequality of the Hölder-McCarty inequality. Furthermore we extend it to the case of two variables using the operator mean in the Kubo-Ando theory. Consequently we have a noncommutative version of the Greub-Rheinboldt inequality as an extension of the Kantrovich one. Finally we discuss about order preserving properties of increasing functions through the Kantorovich inequality.

1. Introduction. In [1], we proved the covariance-variance inequality in the noncommutative probability theory established by Umegaki[12]:

$$|\mathrm{Cov}(A,B)|^2 \le \mathrm{Var}(A)\mathrm{Var}(B), \tag{1}$$

where $\mathrm{Cov}(A,B)$ and $\mathrm{Var}(A)$ are defined as

$$\mathrm{Cov}(A,B) = (B^*Ax,x) - (B^*x,x)(Ax,x) \text{ and } \mathrm{Var}(A) = \mathrm{Cov}(A,A)$$

for (bounded linear) operators A, B acting on a Hilbert space H and a fixed unit vector $x \in H$.

The covariance-variance inequality has many applications for operator inequalities, see [1,2,6]. Among others, we pointed out that (1) implies the celebrated Kantorovich inequality: If a positive operator A on a Hilbert space H satisfies $0 < m \le A \le M$, then for each unit vector $x \in H$

$$(Ax,x)(A^{-1}x,x) \le \frac{(m+M)^2}{4mM}, \tag{2}$$

or equivalently,

$$(A^2x,x) \le \frac{(m+M)^2}{4mM}(Ax,x)^2. \tag{3}$$

Since the covariance-variance inequality is equivalent to the Cauchy-Schwarz inequality, the Kantorovich inequality lies on the line of the Cauchy-Schwarz inequality. More precisely, it is considered as an estimation of the ratio of factors appearing in the Cauchy-Schwarz inequality. Another viewpoint is to estimate the difference of the factors. Actually it has been done in the numerical case. Its operator version will be given by the covariance-variance inequality in the below.

On the other hand, the Hölder-McCarthy inequality[3,8] is a generalization of the Cauchy-Schwarz inequality. Along with our argument, we attempt to generalize the Hölder-McCarthy

---

1991 Mathematics Subject Classification. Primary 47A30, 47A63 and Secondary 46L89.
Key words and phrases. covariance of operator, variance of operator, noncommutative probability, Kantorovich inequality, Cauchy-Schwarz inequality, Hölder-McCarthy inequality, condition number.