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MARCUS–THOMPSON TYPE THEOREMS FOR HILBERT SPACE OPERATORS

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ABSTRACT. Marcus and Thompson proved that the spectrum of the Hadamard product of normal matrices is contained in the polygon spanned by the products of eigenvalues of the matrices. We give it an extension to hyponormal operators on a Hilbert space by virtue of a recently established extension of the Toyama-Marcus-Khan theorem due to J.I.Fujii.

1. Introduction. The Hadamard product $A * B$, the entrywise product of matrices A and B, is studied in detail. T.Ando [1] gave a beautiful perspective on inequalities involving Hadamard product : For operators A and B acting on a Hilbert space H, being expressed as infinite matrices for a fixed orthonormal base $\{e_n\}$, the Hadamard product $A * B$ is also defined by the entrywise product, however it is less studied than matrix case.

Main difference of them lies in the fact that the Hadamard product $A * B$ of matrices A and B obtained by filtering the tensor product $A \otimes B$ through a positive contractive linear mapping, being assured by the Toyama-Marcus-Khan theorem. Unfortunately, there was no corresponding theorem for operators.

Very recently, J.I.Fujii [8] gives an elegant constructive proof to the theorem for operators (Theorem B in the below), and he extends Ando's inequalities for operators, cf. [2]. In [10], we give also certain comments on it.

In the present note, we extend the Marcus-Thompson theorem for normal matrices to hyponormal operators (Theorem 4). In connection with graph theory and characters of operator algebras, we also observe around that theorem.

2. The Marcus-Thompson theorem. In the below, we follow mainly notations and terminologies due to Halmos [13]. An operator means a bounded linear operator acting on a Hilbert space. The spectrum, numerical range and closed numerical range of an operator A are denoted by $\sigma(A), W(A)$ and $\overline{W}(A)$ respectively. The convex and closed convex hull of a subset X in the plane are denoted by co X and $\overline{\text{co}}\ X$ respectively. For subsets X and Y in the plane, XY stands for the subset $\{xy\ ;\ x \in X, y \in Y\}$.

We first cite the Marcus-Thompson theorem [15].

Theorem A. *Let A and B be normal matrices with eigenvalues $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ respectively. Then the eigenvalues of the Hadamard product $A * B$ lie in the convex polygon in $\mathbb{C}$ supported by $\{a_i b_j\ ;\ i, j = 1, 2, \cdots, n\}$.*

For the sake of convenience, we rephrase it as follows :

Theorem A'. *Let A and B be normal operators on a finite dimensional Hilbert space. Then $\sigma(A * B)$ is contained in co $\sigma(A)\sigma(B)$.*

Now, to extend Theorem A to that of operators, an operator version of the Toyama-Marcus-Khan theorem due to J.I.Fujii [8] is quite useful. For a fixed orthonormal base, it is done by the following way.

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