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A note on the differentiability of the distance function to regular submanifolds of Riemannian manifolds

Kazuko MATSUMOTO

Introduction. Let M be a C^∞ Riemannian manifold with a metric g, let S be a submanifold of M and denote by $d(x)$ the distance from $x \in M$ to S induced by the metric g. In the study of various problems of analysis, the function $d = d(x)$ is a useful tool and one must ensure that it is sufficiently differentiable (on some open subset of M) for one's purpose.

In this paper we prove that if S is a C^k regular submanifold of M and $2 \leq k \leq \infty$, then there exists an open subset Δ of M such that $S \subset \Delta$ and the function $h = h(x) = d(x)^2$ is of class C^k on Δ. Here we say that S is a C^k regular submanifold of M if each point x_0 of S has a C^k coordinate neighborhood (U, ψ), $\psi = (\psi_1, \dots, \psi_n)$, such that $S \cap U = \{p \in U : \psi_{r+1}(p) = \cdots = \psi_n(p) = 0\}$, where $n = \dim M$ and $0 \leq r \leq n-1$. In particular, the set S has no boundary but it needs not be closed or connected.

When S is a hypersurface of the Euclidean space $\mathbf{R}^n$, it is easy by the implicit function theorem to see that if S is of class C^k, $k \geq 2$, then there exists an open set including S where h is of class C^{k-1}. In this case, Gilbarg-Trudinger ([2], Lemma 1 of Appendix) showed, as the strict result of Serrin ([5], Lemma 1 of Chapter I, §3), that h is further of class C^k on some open set including S. Their proofs depend on the geometric method, but later Krantz-Parks [3] showed it by elementary means (see also Krantz [4], pp. 136–137). Our proof in this paper is the extension of Krantz-Parks' one.

We note here that the statement above is false in the case $k = 1$. In fact, there is a C^1 curve S in the Euclidean space $\mathbf{R}^2$ which contains a point without positive reach (see, for example, [3]). It follows from the general result of Federer ([1], Theorem 4.8) that the function $h = d^2$ is then not differentiable near the point of S without positive reach.