Nihonkai Math. J.
Vol.14(2003), 95-98

CHARACTERIZATION OF COMPLEX SPACE FORMS IN TERMS OF GEODESICS AND CIRCLES ON THEIR GEODESIC SPHERES

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ABSTRACT. In this paper we pay particular attention to geodesics and circles on geodesic spheres in a given Kähler manifold. We characterize complex space forms in the class of Kähler manifolds from this point of view.

1. Introduction.

In this paper we characterize complex space forms among Kähler manifolds by observing the structure torsion of *geodesics* on their geodesic spheres. A complex n-dimensional complex space form is a Kähler manifold of constant holomorphic sectional curvature c, which is locally congruent to either a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n(c)$, according as c is positive, zero or negative. Among real hypersurfaces in a complex space form geodesic spheres have many nice properties. For a unit tangent vector $v \in T_xN$ of a real hypersurface N in a Kähler manifold M, we put $\eta(v) = \langle v, \xi_x\rangle$ and call its structure torsion. Here ξ is the characteristic vector field of N in M which is defined by $\xi = -J\mathcal{N}$ with unit normal vector field $\mathcal{N}$ and complex structure J of M. For a geodesic γ on N which is parameterized by its arc-length, we can define a structure torsion function η_γ by $\eta(\dot{\gamma})$. When N is a geodesic sphere in a complex space form, the structure torsion η_γ for an arbitrary geodesic γ is a constant function. Our main result in this paper is Theorem 1 which characterizes complex space forms among Kähler manifolds by this property. We also give a characterization of complex Euclidean spaces by the extrinsic shape of circles on geodesic spheres (Theorem 2).

2. Characterization of complex space forms.

For a Riemannian manifold $(M, \langle\ ,\ \rangle)$ of dimension greater than 2, we denote by $G_x(r)$ a geodesic sphere of radius r centered at $x \in M$, and by $A = A_{x,r}$ the shape operator of $G_x(r)$ in M with respect to the outward unit normal vector field $\mathcal{N}$. We then have the following relationship between the Riemannian connections $\widetilde{\nabla}$ of M and ∇ of $G_x(r)$:

$$\text{(2.1)} \qquad \widetilde{\nabla}_X Y = \nabla_X Y + \langle A_{x,r}X, Y\rangle\mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_X\mathcal{N} = -A_{x,r}X.$$

2000 *Mathematics Subject Classification.* Primary 53B25, Secondary 53C40.

Key words and phrases. complex space forms, Kähler manifolds, geodesic spheres, geodesics, structure torsion, circles, first curvature.

The first author partially supported by Grant-in-Aid for Scientific Research (C) (No. 14540075), Ministry of Education, Science, Sports and Culture.

The second author partially supported by Grant-in-Aid for Scientific Research (C) (No. 14540080), Ministry of Education, Science, Sports and Culture.