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Weak Projections on Unital Commutative C^*-Algebras

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1. Preliminary

Let Ω be a compact Hausdorff space and let $C(\Omega)$ be the space of complex valued continuous functions on Ω. With the supremum norm, $C(\Omega)$ is a unital commutative C^*-algebra. Let S be a unital C^*-subalgebra of $C(\Omega)$. A bounded linear operator P on $C(\Omega)$ is called a *projection* onto S if $Ph = h$ for every $h \in S$ and the range of P equals to S. A bounded linear operator Q on $C(\Omega)$ is called a *weak projection* for S if $Qh = h$ for every $h \in S$. If P is a projection onto S, then P is a weak projection for S. Converse of this assertion is not true. A counterexample is $S = \{f \in C([0,1]); f(1/3) = f(x)$ for $1/3 \leq x \leq 2/3\}$. For a unital C^*-subalgebra S of $C(\Omega)$, there may not exist a weak projection for S. Our problem in this paper is to find which conditions on S there exists a weak projection for S.

A motivation of this study comes from Korovkin type approximation theorems. A subset E of $C(\Omega)$ is called a *Korovkin set* if for every sequence of bounded linear operators $\{T_n\}_n$ on $C(\Omega)$ such that $\|T_n\| \leq 1$ for every n and $T_n h \to h$ for each $h \in E$, it holds $T_n f \to f$ for every $f \in C(\Omega)$. Korovkin [4] (see also [6]) proved that $\{1, x, x^2\}$ is a Korovkin set of $C([0,1])$. There are many researches on Korovkin type approximation theorems, see [1, 3, 5].

By the definitions, if S is a unital C^*-subalgebra of $C(\Omega)$ and S is a Korovkin set, then there are no weak projections Q for S such that $Q \neq I$ and $\|Q\| = 1$.

Let S be a unital C^*-subalgebra of $C(\Omega)$. For $x \in \Omega$, put

$$E(x) = \{y \in \Omega; f(y) = f(x) \text{ for every } f \in S\}.$$

Then $E(x)$ is a closed subset of Ω, and it holds $E(x) = E(y)$ or $E(x) \cap E(y) = \emptyset$. We call the family $\{E(x)\}_{x\in\Omega}$ the Shilov decomposition for S. We have the following proposition.

Proposition. *Let S be a unital C^*-subalgebra of $C(\Omega)$ and let $\{E(x)\}_{x\in\Omega}$ be the Shilov decomposition for S. Assume that there exist a non-empty open subset U of Ω and a continuous map φ from U to Ω such that*

i) $\varphi(x) \in E(x)$ *for* $x \in U$,

ii) $\varphi(x) \neq x$ *for* $x \in U$.

Then there exists a weak projection Q for S such that $Q \neq I$ and $\|Q\| = 1$.

Proof. Let φ be a continuous map satisfying i) and ii). We shall prove the existence of a weak projection Q for S with $Q \neq I$ and $\|Q\| = 1$. Take a point x_0 in U and a continuous