Reflexivities with modifiers and conditions for reflexivity of Banach spaces

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1. Reflexivities with modifiers

Denote by X' and X'' the first conjugate spece and the second one of a normed space X respectively. And for a subset A of X' let ${}^{\circ}A$ and A° denote the annihilator of A in X and in X'' respectively. We shall use briefly the symbol $A^{\circ\circ}$ in place of $(A^{\circ})^{\circ}$.

Let F be a subset of X' and consider the linear operator $\pi_F=\mu\cdot\pi : X \longrightarrow X''/F^{\circ}$ where $\pi : X \longrightarrow X''$ is the canonical imbedding and $\mu : X'' \longrightarrow X''/F^{\circ}$ is the natural operator, X''/F° being the quotient normed space defined in the usual way. It is a problem to find out a subset F of X' such that the operator π_F is surjective, namely $\pi_F(X)=X''/F^{\circ}$. When such a subset F exists in X', we shall call, for a time, X to be *reflexive with the modifier* F°, or shortly *reflexive (mod.* F°*)*. A trivial example of such a subset is the set (0) consisting of the zero element 0 of X' alone, for $(0)^{\circ}=X''$. If F_1 and F_2 are two subsets of X' such that $F_1 \supset F_2$, we have $F_1{}^{\circ} \subset F_2{}^{\circ}$, hence if X is reflexive (mod. $F_1{}^{\circ}$) it is so also with the modifier $F_2{}^{\circ}$. A maximal subset F of X' for which X is reflexive (mod. F°) may be, in general, a proper subset of X'. Since $(X')^{\circ}=(0)$, X is reflexive (mod. $(X')^{\circ}$) if and only if X is reflexive in the usual sense. It is easy to prove the following

THEOREM 1. *X' is reflexive (mod. $(\overline{\pi(X)})^{\circ}$), where $\overline{\pi(X)}$ means the closure with respect to the norm topology of X''. And $X'''=\pi_1(X') \oplus (\overline{\pi(X)})^{\circ}$ where $\pi_1 : X' \longrightarrow X'''$ is the canonical imbedding.*

PROOF. For any $x''' \in X'''$ define $x' \in X'$ by $x'x=x'''(\pi x)$ for each $x \in X$. Then we have $x'''-\pi_1 x' \in (\pi(X))^{\circ}$. Since $(\overline{\pi(X)})^{\circ}=(\pi(X))^{\circ}$ and $\pi_1(X') \cap (\overline{\pi(X)})^{\circ}=(0)$, the proof is complete.

This theorem implies, as a special case, that X' is reflexive if X is reflexive. J. Dixmier [2, Theorem 15] has shown that $X'''=\pi_1(X') \oplus (\pi(X))^{\circ}$ where X is a Banach space. This is also a special case of our Theorem 1, for $\pi(X)$ is closed if X is complete.

Since $(\mathrm{v}F)^{\circ}=F^{\circ}$, where $\mathrm{v}F$ means the closed linear subspace generated by F, we may assume, in this problem, that F is a closed linear subspace of X', without loss of generality.

The following theorem follows from the second part of Theorem 1.