A Banach algebra which is an ideal in the second conjugate space II

By

Seiji WATANABE

(Received May 20, 1975)

1. Introduction

Let A be a Banach algebra, A^{**} its second conjugate space. Then A^{**} becomes a Banach algebra under the Arens multiplications. For any Banach space X, let π be the cononical embedding of X into X^{**}. When does A^{**} contain $\pi(A)$ as an ideal? In [5] we investigated the condition under which $\pi(A)$ is an ideal in A^{**}. Here we shall consider the following problem.

(1) When is $\pi(A)$ a two-sided ideal in A^{**}?

(2) When is $\pi(A)$ a block subalgebra in A^{**}?

i. e. $\pi(A)A^{**}\pi(A)\subset\pi(A)$.

If $\pi(A)$ is an ideal in A^{**}, it is a block subalgebra of A^{**}.

A Banach algebra A is called *weakly compact* if every left and right multiplication operators on A are weakly compact.

In [5] we have shown that $\pi(A)$ is an ideal in A^{**} if and only if A is weakly compact.

In §3 we shall investigate the special case, and obtain an improvement of a result in [5].

We shall use the notations and definitions given in [5] without notice.

2. General case

Let A be a Banach algebra. Denote by L_a (resp. R_a) the left (resp. right) multiplication operator on A.

Then we have

$$L_a^*(f)=f\circ a,\quad R_a^*(f)=a*f,\quad L_a^{**}(F)=\pi(a)\circ F$$

$$R_a^{**}(F)=F*\pi(a)\quad (a\in A, f\in A^*, F\in A^{**})$$

where T^* (resp. T^{**}) denote the conjugate (resp. second conjugate) operator of an operator T.

Hence we have the following two theorems from the well-known result on weakly compact operators [see 2].