A Banach algebra which is an ideal in the second dual space

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1. Introduction

The second dual space A^{**} of a Banach algebra A can also be considered as a Banach algebra by the use of Arens muliplication [Arens, 1]. When A is embedded in A^{**} by the canonical mapping, A is only a subalgabra of A^{**} but is not an ideal in A^{**} in general. When does A^{**} contain A as an ideal? It is well-known that when A is a C*-algebra, A is a dual C*-algebra if and only if A is a two-sided ideal in A^{**}. Recently many authors obtained other characterizations in this case. In this paper, we shall consider the above problem for general Banach algebras. In §2, we shall show that a Banach algebra which is an ideal in the second dual space is characterized by the weak compactness of left or right multiplications on A. In §3, we shall show that for a group algebra A of a locally compact topological group G, A is a two-sided ideal in A^{**} if and only if G is compact. Moreover we shall show analogous result for a certain subalgebra in A^{**}.

2. Preliminaries

Let A be a Banach algebra. Denote by A^* the dual space of A, and denote by A^{**} the second dual space of A. Throughout we denote by π the canonical embedding of A into A^{**}. Let $x, y \in A, f \in A^*$ and $F, G \in A^{**}$. Then we define the following functions:

$$(f, x) \longrightarrow f \circ x : A^* \times A \longrightarrow A^* \quad \text{where } (f \circ x)(y) = f(xy),$$

$$(F, f) \longrightarrow F \circ f : A^{**} \times A^* \longrightarrow A^* \quad \text{where } (F \circ f)(x) = F(f \circ x),$$

and

$$(F, G) \longrightarrow F \circ G : A^{**} \times A^{**} \longrightarrow A^{**} \quad \text{where } (F \circ G)(f) = F(G \circ f).$$