An elementary proof of Gleason-Kahane-Zelazko's theorem for complex Banach algebra with a hermitian involution

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1. Introduction

Gleason [2], Kahane and Zelazko [3] proved independently the following;

THEOREM (Gleason-Kahane-Zelazko). *Let A be a complex unital Banach algebra and let f be a linear functional on A. Then f is multiplicative if and only if $f(a)\in Sp(a)$ $(a\in A)$.*

Their proof is based on Hadamard's factorization theorem. By Choda and Nakamura [1], an elementary proof of this theorem for B*-algebra was presented without depending on such a theorem from the theory of functions.

The purpose of our paper is to present an elementary proof of this theorem for a complex Banach algebra with a hermitian involution. Throughout this paper, we use the standard notations and terminologies from [4].

2. The main theorem

LEMMA. *Let A be a complex Banach algebra with a hermitian involution and let f be a linear functional on A.*
If $f(a)\in Sp_A(a)$ $(a\in A)$, then we have

$$f(xh)=f(x)f(h) \quad (x\in A,\ h\in A_h),$$

where A_h denotes the set of all self-adjoint elements of A.

PROOF. We shall suppose, without loss of generality, that A possesses an identity element 1.

Let $k\in A_h$ be such that $f(k)=0$, and B be a maximal commutative $*$-subalgebra of A which contains 1 and k, and Φ_B be the carrier space of B. Then we get

$$\mathrm{Sp}_A(x)=\mathrm{Sp}_B(x) \quad (x\in B).$$

Since $k^2+ik\in B$ and $f(k^2+ik)=f(k^2)$, there exists, from our assumption, an element $\Psi\in\Phi_B$ such that