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INVERSE PROBLEMS IN FOURIER ANALYSIS AND SUMMABILITY THEORY*

MARK A. PINSKY†

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1. Introduction. The oldest theorems of Fourier analysis give sufficient conditions under which

$$\lim_{M\to\infty} S_M f(x) = \frac{1}{2}[f(x+0)+f(x-0)]. \tag{1}$$

Here f is an integrable function on the circle and $S_M f$ is the associated Fourier partial sum. This can be viewed as an *inverse problem*, namely to determine the local average of a function, given the knowledge of its Fourier partial sums. These conditions can be substantially relaxed if the Fourier partial sum is replaced by a suitable summability method.

One can also ask to obtain the jump from the Fourier partial sums, for example by studying the derivative $(S_M f)'(x)$. This inverse problem is more delicate, since the localization principle of Riemann is not valid for the derivatives of the partial sums. Nevertheless in 1913 Fejér [F] found that, assuming suitable regularity

$$\lim_{M\to\infty} \frac{(S_M f)'(x)}{M} = C_1[f(x+0)-f(x-0)] \tag{2}$$

where C_1 is a universal constant, whose value depends on the convention used to define $S_M f$. Results of this type also hold for various summability procedures and were studied by Lukács [L], Zygmund [Z] and others. In all of these works the jump in (2) may be replaced by a suitable local *average jump* whose precise definition depends on the regularity of the summability method.

Instead of using the derivative to retrieve the jump, one may also use the conjugate partial sum $\tilde{S}_M f$. In this case one obtains a logarithmic behavior, leading to a result of the form

$$\lim_{M\to\infty} \frac{(\tilde{S}_M f)(x)}{\log M} = C_2[f(x+0)-f(x-0)] \tag{3}$$

for another universal constant. Results of this type were obtained by Fejér [F] and Lukács [L] for the Fourier partial sum and by Móricz[M2] for Abel summability of Fourier series on the circle.

The purpose of this paper is to give a unified treatment of these results for functions on the line. The corresponding results on the circle can be obtained by periodization techniques. When we come to the analysis of the conjugate function, it is most efficient to use the definition of the conjugate Poisson integral formulated by Koosis[Ko], which is simultaneously defined on all of the Lebesgue spaces $L^p(R), 1\le p\le\infty$.

*Received March 16, 2004; accepted for publication September 1, 2004.
†Mathematics Department, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208-2730, USA (pinsky@math.northwestern.edu).