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MULTIVALUED DIFFERENTIAL EQUATIONS ON CLOSED SETS

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Abstract. Let X be a Banach space, $D \subset X$ closed, $J = [0,a] \subset \mathbf{R}$ and $F : J \times D \to 2^X \backslash \{\emptyset\}$ a multivalued map. We consider the initial value problem

$$u' \in F(t,u) \quad \text{a.e. on } J,\ u(0) = x_0 \in D. \tag{1}$$

Solutions of (1) are understood to be a.e. differentiable with $u' \in L^1_X(J)$ such that $u(t) = x_0 + \int_0^t u'(s)\,ds$ on J and (1) is satisfied. We first give sufficient conditions for existence of solutions in the autonomous case $F(t,x) = F(x)$ and indicate relations to the fixed problem $x \in F(x)$. We concentrate on upper semicontinuous F with compact convex images and give counter-examples if F is lower semicontinuous or the $F(x)$ are not convex. Then the time-dependent problem (1) is considered by reduction to the autonomous case. Finally we prove existence of solutions which are monotone with respect to a preorder or Lyapaunov-like functions. Since we allow $\dim X = \infty$, we find it necessary to exploit Zorn's Lemma. As a by-product of this approach we achieve considerable simplification of proofs or improvement of some results known for $\dim X < \infty$.

1. Preliminaries. All concepts not discussed in detail in the sequel can be found in several places, e.g. in [5], [6]. First of all, it is easy to see that a necessary condition for $u' \in F(u)$, $u(0) = x_0$ to have a solution for every $x_0 \in D$ is given by $F(x) \cap T_D(x) \neq \emptyset$ on D where

$$T_D(x) = \{y \in X : \varliminf_{\lambda \to 0+} \lambda^{-1}\rho(x + \lambda y\,,\, D) = 0\} \text{ with } \rho(z,D) = \inf_D |z - x|. \tag{2}$$

In case D has nonempty interior $\overset{0}{D}$ this is only a condition at the boundary ∂D, since $T_D(x) = X$ for $x \in \overset{0}{D}$. If D is closed and convex we have $T_D(x) = \overline{\{\lambda(y - x) : \lambda \geq 0\,,\, y \in D\}}$.

In the majority of applications the "multis" F are *upper semicontinuous* (*usc*), i.e., $F^{-1}(A) = \{x \in D : F(x) \cap A \neq \emptyset\}$ is closed whenever $A \subset X$. F is said to be *lower semicontinuous* (*lsc*), if $F^{-1}(V)$ is open whenever $V \subset X$ is, and *continuous* if F is continuous with respect to the Hausdorff metric. Since our main interest is in $\dim X = \infty$

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