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ON $\mathbf{x}' = \mathbf{f}(\mathbf{t}, \mathbf{x})$ AND HENSTOCK–KURZWEIL INTEGRALS

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Abstract. The existence and continuous dependence on a parameter theorem for $x' = f(t,x)$ by using the Henstock-Kurzweil integral setting are proved.

1. Introduction. There has been a resurgence of interest in integrals of highly oscillating functions [4, 8, 9, 13, 16, 18] and their applications to differential equations recently [1, 10, 17]. Let $F(x) = x^2 \sin x^{-2}$ if $x \neq 0$ and $F(0) = 0$, then $F'(x)$ exists on $[0,1]$. Hence, $F'(x)$ is Newton integrable there. The function $F'(x)$ is highly oscillating and not Lebesgue integrable there. The Henstock-Kurzweil integral integrates highly oscillating functions and encompasses Newton, Riemann and Lebesgue integrals [7, 8 p. 35-38, 13]. This integral was introduced by Kurzweil and Henstock independently in 1957/58 [7, 11]. Kurzweil defined his integral mainly for applications to differential equations [5, 11]. It has been shown that the Henstock-Kurzweil integral is equivalent to the Denjoy-Perron integral [6, 8, 11, 13, 19]; in particular, the equivalence of the Henstock-Kurzweil to the Perron integral was first shown in the paper [11] of J. Kurzweil.

A major part of the classical theory of differential equations is concerned with the existence of solution of $x' = f(t,x)$ with $x(\tau) = \xi$. Cauchy and Peano proved an existence theorem by assuming the continuity of f, whereas Carathéodory proved it by using Lebesgue integrals [2, 3]. However, if there exists a solution $x(t)$ of $x' = f(t,x)$, such that $x'(t) = f(t, x(t))$ for every t in some interval $[a,b]$, then $f(t,x(t))$ is Newton integrable in this interval. For example, if $f(t,x) = t^2 x + F'(t)$, where $F(t)$ is given at the beginning of this section, then the solution of $x' = f(t,x)$ is $x(t) = e^{t^3/3} \int_0^t e^{-s^3/3} F'(s)ds$ with $x(0) = 0$. Cauchy-Peano's and Carathéodory's existence theorems do not include this case because F' is not integrable in the Lebesgue sense. It is therefore natural to consider the differential equation $x' = f(t,x)$ by using the Newton integral setting or the Henstock-Kurzweil integral setting. The latter includes the former as well as Lebesgue's.

In this note, we shall generalize Carathéodory's existence theorem on a solution of $x' = f(t,x)$ and prove a theorem on continuous dependence on a parameter by using the Henstock-Kurzweil integral setting.

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