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# AN EXISTENCE THEOREM FOR A PARABOLIC FREE BOUNDARY PROBLEM

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**Abstract.** In this paper, the existence of a (classical) solution to a diffusion process with free boundary is obtained by applying Rothe's method (method of lines). A collection of semi-discrete approximate problems is obtained by discretizing the partial differential equations with respect to the time variable. A sequence of approximate solutions to the diffusion process is constructed from this collection of semi-discrete problems by interpolation between the lines in the grid. This sequence is then shown to converge to a solution to the free boundary problem.

**1. Introduction.** In an earlier article [6], we investigated the convergence of the method of lines approximations to the following implicit nonlinear parabolic free boundary problem:

$$\begin{cases} u_{xx} = u_t + f(t,x,u,u_x), \quad 0 < t \le T, \ 0 < x < s(t), \\ (Bu)(t) = \alpha(t), \quad u(t,s(t)) = u_x(t,s(t)) = 0, \quad 0 < t \le T, \\ u(0,0) = 0, \quad s(0) = 0, \end{cases} \tag{P}$$

where $(Bu)(t) := au(t,0) - bu_x(t,0)$, with $a, b \ge 0$, $a + b = 1$, and $\alpha(t) > 0$ for $0 < t \le T$. In this paper, we extend these earlier results to give a constructive existence proof for problem (P). Our approach to this problem is to apply the transversal line method (also known as Rothe's method [3], [4]) and is motivated, in part, by some results of G.H. Meyer ([1], [2]) on a similar problem.

In the transversal line method, a finite difference is substituted in the place of the time derivative $u_t$ in (P). Using the notation $h := T/N$, where $N$ is a whole number, $N \ge 1$, $t_n := nh$, $u_n(x) \approx u(t_n, x)$ and $\delta u_n(x) := h^{-1}(u_n - u_{n-1})$, $n = 1, \ldots, N$, we may associate with the above problem a discretized problem

$$\begin{cases} u_n'' = \delta u_n + f(t_n, x, u_n, u_n'), \quad 0 \le x \le s_n, \\ Bu_n = \alpha_n, \quad u_n(s_n) = u_n'(s_n) = 0, \\ u_0(x) \equiv 0, \quad s_0 = 0, \end{cases} \tag{$\mathrm{P}_N$}$$

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