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A SPECTRAL THEOREM FOR REVERSIBLE SECOND ORDER EQUATIONS WITH PERIODIC COEFFICIENTS*

Daniel C. Offin

Department of Mathematics and Statistics, Queens University, Kingston, Canada K7L 3N6

(Submitted by: A.R. Aftabizadeh)

Abstract. Consider the second order linear differential operator $L = -\frac{d^2}{dt^2} + Q(t)$, where $Q(t)$ is symmetric $n \times n$ matrix and T-periodic. We are concerned in this paper with the relationship between the spectrum of L as a differential operator and the spectrum of the monodromy matrix of the system of equations $Lx = 0$. We prove that if the system is reversible, $Q(t) = Q(-t)$, and the (Morse) index $L = 0$ then the monodromy matrix has no eigenvalues on the unit circle. We use techniques from symplectic geometry and in particular an antisymplectic involution to prove this result.

0. Introduction. Consider the second order linear differential operator

$$L = -\frac{d^2}{dt^2} + Q(t), \quad Q(t+T) = Q(t) \in \text{Sym}\,(\mathbb{R}^n).$$

This operator is essentially self adjoint on the Sobolev space H_T^1 with periodic boundary conditions

$$H_T^1 = \{x\colon [0,T] \to \mathbb{R}^n : x \in \text{AC}[0,T],\ x' \in L^2,\ x(0) = x(T)\},$$

equipped with the sum of the L^2 scalar product on arcs x and their time derivatives x'. The (Morse) index for the operator L is the number (finite) of negative eigenvalues of L. The question we will address in this paper is whether the Morse index equal to zero implies that the homogeneous equations $Lx = 0$ have an exponential dichotomy. This question, raised by Poincare and later in an early work of Birkhoff [2] (p130) concerning closed geodesics of minimal length, was answered in the affirmative by Hartman [9] in case $Q > 0$ as a quadratic form. A different approach to this case utilizing Sturmian comparison theorems may be found in Arnold [1]. To see the connection with our question concerning index theory, we outline this background. The homogeneous equation $Lx = 0$ is the Euler equation for the quadratic functional

$$A_L(x) = \int_0^T \{\frac{1}{2}x'^2 + x^*Qx\}\,dt$$

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