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BEST CONSTANT FOR THE EMBEDDING OF THE SPACE $H^2 \cap H_0^1(\Omega)$ INTO $L^{2N/(N-4)}(\Omega)$

R.C.A.M. VAN DER VORST†
Mathematical Institute, Leiden University, The Netherlands

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1. Introduction. In the study of fourth order elliptic equations in bounded domains [16], it is important to know the constant of the continuous, noncompact embedding of the space $H^2 \cap H_0^1$ into $L^{2N/(N-4)}$,

$$H^2 \cap H_0^1(\Omega) \to L^{\frac{2N}{N-4}}(\Omega).$$

This specific embedding plays a role when fourth order problems with boundary conditions $u = \Delta u = 0$ on $\partial\Omega$ are considered. Here Ω is a smooth, bounded domain in $\mathbb{R}^N$. The main objective of this paper is to solve the following problem.

Problem (I). *Find the largest constant K_1 for which the inequality*

$$\|u\|_{2N/(N-4)} \leq K_1^{-1/2}\|u\|_{2,2}, \quad \forall u \in H^2 \cap H_0^1(\Omega), \tag{1.1}$$

is valid.

The norms used here are defined by

$$\|u\|_{2N/(N-4)} = \left(\int_\Omega |u|^{\frac{2N}{N-4}}\,dx\right)^{\frac{N-4}{2N}}, \quad \|u\|_{2,2} = \left(\int_\Omega |\Delta u|^2\,dx\right)^{1/2}.$$

The analog of Problem (I) for the case $\Omega = \mathbb{R}^N$ can be answered and the largest constant possible is K_0 which is given by

$$K_0 = \min\left\{\int_{\mathbb{R}^N} |\Delta u|^2\,dx : \ u \in \mathcal{D}^{2,2}(\mathbb{R}^N), \ \int_{\mathbb{R}^n} |u|^{\frac{2N}{N-4}}\,dx = 1\right\}, \tag{1.2}$$

where the space $\mathcal{D}^{2,2}(\mathbb{R}^N)$ is the completion of $\mathcal{D}(\mathbb{R}^N)$ in the norm $\|\cdot\|_{2,2}$. This was studied by Lions [8]. He also proved that there exists a minimizer for (1.2) which is uniquely determined up to translations and dilations. The explicit form of this minimizer can be found in [5], [7] and is given by

$$U_{\epsilon,x_0}(x) = C_N \frac{\epsilon^{\frac{N-4}{2}}}{(|x-x_0|^2+\epsilon^2)^{\frac{N-4}{2}}}, \tag{1.3}$$

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