Differential and Integral Equations, Volume 7, Number 5, September 1994, pp. 1215–1225.

SUB AND SUPERSOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS ON ALL OF $\mathbb{R}^n$

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This paper is dedicated to the memory of Peter Hess

Abstract. Sub and supersolutions are constructed for the semilinear elliptic equation $-\Delta u = \lambda g(x) f(u)$ on all of $\mathbb{R}^n$ which arises in population genetics. It is shown that the theory of existence of solutions is very different in the case $n = 1$ or 2 and in the case $n \geq 3$.

1. Introduction. In this paper we shall discuss the construction of sub- and supersolutions as well as the existence and nonexistence of solutions of the equation

$$-\Delta u = \lambda g(x) f(u), \quad 0 < u < 1, \quad x \in \mathbb{R}^n, \tag{1.1$_\lambda$}$$

which arises in population genetics (see [1, 3]). The unknown function u corresponds to the relative frequency of an allele and is hence constrained to have values between 0 and 1. The real parameter $\lambda > 0$ corresponds to the reciprocal of a diffusion coefficient so that the case of small λ considered in this paper corresponds to diffusion being large.

We assume throughout that g satisfies

$$(G) \qquad \begin{array}{l} g : \mathbb{R}^n \to \mathbb{R} \text{ is smooth such that } g(x_0) > 0 \text{ for some } x_0 \in \mathbb{R}^n \\ \text{and there exists } R_0 > 0 \text{ such that } g(x) < 0 \text{ whenever } |x| > R_0. \end{array}$$

This assumption corresponds to the fact that an allele has an advantage at some points x in $\mathbb{R}^n$ (where $g(x) > 0$), but is disadvantaged for $|x| > R_0$.

In the population genetics model f is considered to be the cubic function $f(u) = u(1-u)[h(1-u) + (1-h)u]$, for some constant h, $0 < h < 1$. We shall assume throughout that f satisfies the condition

$$(F) \qquad \begin{array}{l} f : [0, 1] \to \mathbb{R} \text{ is a smooth function such that } f(0) = f(1) = 0, \\ f'(0) > 0, f'(1) < 0, \quad \text{and } f(u) > 0 \text{ for all } 0 < u < 1. \end{array}$$

Received September 1993.

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AMS Subject Classifications: 35J60.