Differential and Integral Equations, Volume 8, Number 3, March 1995, pp. 673 – 679.

A NOTE ON PROBLEMS INVOLVING CRITICAL SOBOLEV EXPONENTS*

D.G. Costa
Department of Mathematical Sciences, University of Nevada, Las Vegas, NV 89154
and
Dept. Matematica, Univ. Brasilia, 70910 Brasilia, Brazil

E.A. Silva
Dept. Matematica, Univ. Fed. Pernambuco, 50.000 Recife, Brazil

(Submitted by: G. Da Prato)

Introduction. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ $(n \geq 3)$ and let $p = 2^* = 2n/(n-2)$ be the limiting exponent in the Sobolev embedding $H_0^1(\Omega) \subset L^p(\Omega)$. In this paper we consider a class of nonlinear elliptic problems of the form

$$-\Delta u = |u|^{p-2}u + f(x,u) \quad \text{in} \quad \Omega,\ u = 0 \ \text{ on } \ \partial\Omega, \tag{P}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying the conditions

$$f(x,u)/u = \lambda + o(1) \ \text{ as } \ u \to 0,$$

$$f(x,u)/|u|^q = o(1) \ \text{ as } \ |u| \to \infty,$$

uniformly for almost every $x \in \Omega$, with $q = p - 1 = (n+2)/(n-2)$ and $\lambda > 0$. Therefore, we can write $(P) = (P)_\lambda$ and

$$f(x,u) = \lambda u + g(x,u), \tag{1}$$

where g satisfies
(f_1) $g(x,u) = o(u)$ as $u \to 0$,
(f_2) $g(x,u) = o(|u|^q)$ as $|u| \to \infty$,
uniformly for almost every $x \in \Omega$.

Since the appearance of the pioneering paper [2] by Brézis-Nirenberg, much attention has been given to such problems with "critical Sobolev exponents" (cf. [1, 3–6, 10], for example). In particular, for the case $g(x,u) \equiv 0$, $n \geq 4$, it was shown in [4] that $(P)_\lambda$ has a nontrivial solution for any $\lambda > 0$. The proof seemed to use the symmetry (oddness) of problem $(P)_\lambda$ in that case. An alternative proof, based on the associated dual functional, was given in [1]. This approach allowed the handling of situations where $g(x,\cdot)$ was increasing and satisfied $(f_1), (f_2)$. Here, we neither assume that $g(x,\cdot)$ is symmetric nor that it is increasing. The parameter $\lambda > 0$ will be taken outside the spectrum $\{\lambda_1, \lambda_2, \dots\}$ of $-\Delta$ on $H_0^1(\Omega)$, say

Received for publication April 1993.
*Research partially supported by CNPq/Brasil.
AMS Subject Classifications: 35J50, 35G30.