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ON PICARD POTENTIALS

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1. Introduction. This paper represents a further contribution to the problem of characterizing all elliptic finite-gap solutions of the stationary Korteweg-de Vries (KdV) hierarchy, a problem posed, e.g., in [22, p. 152]. This theme dates back to a 1940 paper of Ince [15] who studied the Lamé potential

$$q(x) = -s(s+1)\mathcal{P}(x+\omega_3), \quad s \in \mathbb{N},\ x \in \mathbb{R} \tag{1}$$

in connection with the second-order ordinary differential equation

$$\psi''(E,x) + [q(x) - E]\psi(E,x) = 0, \quad E \in \mathbb{C}. \tag{2}$$

Here $\mathcal{P}(x) \equiv \mathcal{P}(x;\omega_1,\omega_3)$ denotes the elliptic Weierstrass function with fundamental periods (f.p.) $2\omega_1$, $2\omega_3$, $\Im(\omega_3/\omega_1) \neq 0$ (see [1], Ch. 18). In the special case where ω_1 is real and ω_3 is purely imaginary the potential q is real-valued and Ince's striking result [15], in modern spectral theoretic terminology, yields the fact that the self-adjoint operator L associated with the differential expression $\frac{d^2}{dx^2} + q$ in $L^2(\mathbb{R})$ has a finite-gap (or finite-band) spectrum of the type

$$\sigma(L) = (-\infty, E_{2s}] \bigcup_{m=1}^{s} [E_{2m-1}, E_{2(m-1)}], \quad E_{2s} < E_{2s-1} < \cdots < E_0. \tag{3}$$

In obvious notation, any potential q that amounts to a finite-gap spectrum of the type (3) is called a finite-gap potential. The proper extension of this notion to complex-valued meromorphic q on the basis of elementary algebro-geometrical concepts is obtained as follows: The starting point is the definition of the so called KdV hierarchy. Let L be the second-order differential expression

$$L = \frac{\partial^2}{\partial x^2} + q,$$

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