Differential and Integral Equations, Volume 3, Number 4, July 1990, pp. 695–708.

SEMICOERCIVE VARIATIONAL PROBLEMS AT RESONANCE: AN ABSTRACT APPROACH*

A. FONDA
SISSA, Strada Costiera, 11, 34014 Trieste, Italy

J.-P. GOSSEZ
Départment de Mathématique, Campus Plaine C.P.214, Université Libre de Bruxelles
Bd du Triomphe, 1050 Bruxelles, Belgium

(Submitted by: Jean Mawhin)

Abstract. We study the coercivity of functionals of the form $a + b$ where a is semicoercive with respect to a subspace and b is coercive on the complementary subspace. Applications are given to the existence of solutions for a semilinear Dirichlet problem.

1. Introduction. This paper is concerned with the existence of solutions to elliptic boundary value problems at resonance with the first eigenvalue. We consider the Dirichlet problem

$$-\mathbf{\Delta} u - \lambda_1 u + g(x,u) = 0 \text{ in } \mathbf{\Omega}, \quad u = 0 \text{ on } \partial\mathbf{\Omega} \tag{P}$$

where $\mathbf{\Omega}$ is a bounded open subset of $\mathbf{R}^N$, and λ_1 the first eigenvalue of $(-\mathbf{\Delta})$ on $\mathbf{H}_0^1(\mathbf{\Omega})$. The Caratheodory function $g : \mathbf{\Omega} \times \mathbf{R} \to \mathbf{R}$ is supposed to satisfy the usual growth condition

$$|g(x,u)| \leq a|u|^{q-1} + b(x)$$

where $q < \infty$ if $N = 2$, $q < 2^* = 2N/(N-2)$ if $N \geq 3$, and where $b(x) \in L^{q'}(\mathbf{\Omega})$, with q' the Hölder conjugate exponent of q; if $N = 1$, it suffices to assume that for any $r > 0$,

$$\sup_{|u| \leq r} |g(x,u)| \in L^1(\mathbf{\Omega}).$$

Under this condition, the associated functional

$$f(u) = \tfrac{1}{2} \int_{\mathbf{\Omega}} \left[|\nabla u|^2 - \lambda_1 |u|^2\right] + \int_{\mathbf{\Omega}} G(x, u(x))\, dx,$$

where $G(x,u) = \int_0^u g(x,s)\, ds$, is a weakly lower semicontinuous $\mathbf{C}^1$ functional on $\mathbf{H}_0^1$ whose critical points are the weak solutions of (P). It follows that if f is coercive

Received March 13, 1989.
*This work was carried out while the first author was visiting the Université Catholique de Louvain
AMS Subject Classifications: 35J20, 35J25.