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Index Theory, Gerbes, and Hamiltonian Quantization

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Abstract: We give an Atiyah-Patodi-Singer index theory construction of the bundle of fermionic Fock spaces parametrized by vector potentials in odd space dimensions and prove that this leads in a simple manner to the known Schwinger terms (Faddeev-Mickelsson cocycle) for the gauge group action. We relate the APS construction to the bundle gerbe approach discussed recently by Carey and Murray, including an explicit computation of the Dixmier-Douady class. An advantage of our method is that it can be applied whenever one has a form of the APS theorem at hand, as in the case of fermions in an external gravitational field.

1. Introduction

There are subtleties in defining the fermionic Fock spaces in the case of chiral (Weyl) fermions in external vector potentials. The difficulty is related to the fact that the splitting of the one particle fermionic Hilbert space H into positive and negative energies is not continuous as a function of the external field. One can easily construct paths in the space of external fields such that at some point on the path a positive energy state dives into the negative energy space (or vice versa). These points are obviously discontinuities in the definition of the space of negative energy states and therefore the fermionic vacua do not form a smooth vector bundle over the space of external fields. This problem does not arise if we have massive fermions in the temporal gauge $A_0 = 0$. In that case there is a mass gap $[-m, m]$ in the spectrum of the Dirac hamiltonians and the polarization to positive and negative energy subspaces is indeed continuous.

If λ is a real number not in the spectrum of the hamiltonian then one can define a bundle of fermionic Fock spaces $\mathcal{F}_{A,\lambda}$ over the set U_λ of external fields A, $\lambda \notin Spec(D_A)$. It turns out that the Fock spaces $\mathcal{F}_{A,\lambda}$ and $\mathcal{F}_{A,\lambda'}$ are naturally isomorphic up to a phase. The phase is related to the arbitrariness in filling the Dirac sea between vacuum levels λ, λ'. In order to compensate this ambiguity one defines a tensor product