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The Estimates of Periodic Potentials in Terms of Effective Masses

E. Korotyaev★

Math. Dept. 2, ETU, 5 Prof. Popov Str., St. Petersburg, 197376, Russia. E-mail: evgeni@niif.spb.su

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Abstract: Let $G_n = (A_n^-, A_n^+)$, $n \geqq 1$, denote the gaps, $M_n^\pm$ be the effective masses and $\Sigma_n = [A_{n-1}^+, A_n^-], A_0^+ = 0$, be the spectral bands of the Hill operator $T = -d^2/dx^2 + V(x)$ in $L^2(\mathbf{R})$, where V is a 1-periodic real potential from $L^2(0,1)$. Let the length gap $L_n = |G_n|$, h_n be the height of the corresponding slit on the quasimomentum domain and $\Delta_n = \pi^2(2n-1) - |\Sigma_n| > 0$ be the band reduction. Let $l_n = \sqrt{A_n^+} - \sqrt{A_n^-}$, $n \geqq 1$, denote the gap length for the operator $\sqrt{T} \geqq 0$. Introduce the sequences $L = \{L_n\}$, $h = \{h_n\}$, $l = \{l_n\}$, $\Delta = \{\Delta_n\}$, $M^\pm = \{M_n^\pm\}$ and the norms $\|f\|_m^2 = \sum_{n>0} (2\pi n)^{2m} f_n^2$, $m \geqq 0$. The following results are obtained: i) The estimates of $\|V\|, \|L\|, \|h\|_1, \|l\|_1, \|\Delta\|$ in terms of $\|M^\pm\|_2$, ii) identities for the Dirichlet integral of quasimomentum and integral of potentials and so on, iii) the generation of i), ii) for more general potentials.

1. Introduction

Let us consider the Hill operator $T = -d^2/dx^2 + V(x)$ in $L^2(\mathbf{R})$, where V is a 1-periodic real potential from $L^1(0,1)$. It is well known that the spectrum of T is absolutely continuous and consists of intervals $\Sigma_1, \Sigma_2, \ldots$. Here $\Sigma_n = [A_{n-1}^+, A_n^-], \ldots, A_{n-1}^+ < A_n^- \leqq A_n^+$, $n \geqq 1$, and let $A_0^+ = 0$. These intervals are separated by the gaps $G_1, G_2, \ldots$, where $G_n = (A_n^-, A_n^+)$. If a gap degenerates, i.e. $G_n = \emptyset$, then the corresponding segments Σ_n, Σ_{n+1} merge. Let $\varphi(x,E), \vartheta(x,E)$ be the solutions of the equation

$$-f'' + Vf = Ef, \quad E \in \mathbf{C}, \tag{1.1}$$

satisfying $\varphi'(0,E) = \vartheta(0,E) = 1$ and $\varphi(0,E) = \vartheta'(0,E) = 0$. We define the Lyapunov function $F(E) = (\varphi'(1,E) + \vartheta(1,E))/2$. The sequence $A_0^+ < A_1^- \leqq A_1^+ < \cdots$ is the spectrum of Eq. (1.1) with the periodic boundary conditions of period 2, i.e. $f(x+2) = f(x)$, $x \in \mathbf{R}$. Here the equality means that $A_n^- = A_n^+$ is the double eigenvalue. We note that $F(A_n^\pm) = (-1)^n$, $n \geqq 1$. The lowest eigenvalue A_0^+ is simple, $F(A_0^+) = 1$, and the corresponding eigenfunction has period 1. The eigenfunctions

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