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Quantum Groups at Odd Roots of Unity and Topological Invariants of 3-Manifolds

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Abstract: Topological invariants of three-manifolds are constructed using quantum groups associated with the A, B, C and D series of Lie algebras at odd roots of unity. These invariants are also explicitly computed for the lens spaces.

1. Introduction

Since the landmark discovery of Jones [1], there has been dramatic progress in the field of knot theory and the theory of 3-manifolds. A distinctive feature of the subject is its close connection with mathematical physics [2]. Indeed, many of the important discoveries are prompted by ideas and techniques from various branches of mathematical physics. For example, an intrinsically three dimensional construction of the Jones and related link polynomials was developed in terms of the quantum Chern–Simons theory [3]; new topological invariants of 3-manifolds were obtained using conformal field theory [4], and quantum groups [5, 6]. It is our aim here to study 3-manifolds using quantum groups [7, 8].

We will apply the construction of Reshetikhin–Turaev [5, 6] to the quantum groups associated with the A, B, C and D series of Lie algebras at odd roots of unity to obtain the corresponding topological invariants of 3-manifolds. We will also compute these invariants for the lens spaces. Previously these quantum groups at even roots of unity were considered by Turaev and Wenzl [9, 6], who made essential use of a result of Kac–Peterson [10] on modular properties of characters of irreducible integrable representations of affine Lie algebras, which does not seem to apply to the odd roots of unity case in any obvious way. The HOMFLY and Kauffman ploynomials associated with these Lie algebras were also used in [20] to construct 3-manifold invariants, which were averages over all possible cablings [19].

The Reshetikhin–Turaev construction makes use of two fundamental theorems in 3-manifold theory, due to Lickorish [11] and Wallace, and Kirby [12] and Craggs respectively. The Lickorish–Wallace theorem states that each framed link in S^3 determines a closed, orientable 3-manifold, and every such 3-manifold is obtainable by surgery along a framed link in S^3. The disadvantage of this description