Yang–Mills Fields on Cylindrical Manifolds and Holomorphic Bundles I

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Abstract: We give complex holomorphic descriptions of Yang–Mills instantons on tubular four manifolds with nontrivial circle bundles over Riemann surfaces as section.

0. Introduction

This is the first of two installments in which we study Yang–Mills instantons, i.e. anti-self-dual connections with finite Yang–Mills energy on tubular manifolds $Y \times R$, where Y is a non-trivial circle bundle over a Riemann surface.

For a circle bundle Y on a Riemann surface Σ, the space $Y \times R$ admits a complex structure and can be compactified to a ruled surface S by, roughly speaking, adding a copy of Σ to each end of the tube $Y \times R$. On the other hand, the finite energy condition and Uhlenbeck's compactness theorem implies that on a manifold with ends any instanton is asymptotic over each end to a flat connection on the section over the end. For a circle bundle Y on a Riemann surface Σ, flat connections on Y can be divided roughly into two classes, those with non-trivial holonomy along the fibre circle as one and those with trivial one as the other one. Accordingly instantons on $Y \times R$ can be divided into three classes, those with flat limits without holonomy along the fibre circle as the first one, those with flat limits with holonomy along the fibre circle as the second one and those with mixed limits as the last one. Roughly speaking, in this first installment, we prove that there is a natural injection of the set of instantons in the first class into the space of holomorphic bundles over S which are flat along the two added divisors. In the next installment, we shall prove that there is a similar injective map from the set of instantons in the second class to the space of certain equivariant holomorphic bundles on a canonical covering of S, branched over the two added divisors, with some preferred filtration and unitary structures in the restrictions over the two divisors. Complex holomorphic description for instantons in the third class can be obtained as a corollary to these two results. For precise statements, see the main text.

To establish our main results, we need to show that given an appropriate in-stanton, there is naturally an associated holomorphic bundle with required properties