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Cut-Down Method in the Inductive Limit Decomposition of Non-Commutative Tori, III: A Complete Answer in 3-Dimension

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Abstract: We use the cut-down method and a multi-dimensional continued fraction approximation to prove that any simple 3-torus is an inductive limit of direct sums of four circle algebras. Consequently, simple 3-tori are classified by the ordered K_0-group with distinguished order unit.

1. Introduction

A non-commutative n-torus is the universal C*-algebra generated by n unitaries, $U_1, U_2, \ldots, U_n$, with non-trivial linear commutation relations

$$(*1) \qquad U_jU_i = (\exp(2\pi i\theta_{ij}))U_iU_j, \quad 1 \leqq i < j \leqq n,$$

where θ_{ij} is a real number. We shall call such unitary generators canonical and a simple non-commutative n-torus a simple n-torus.

Such a C*-algebra is often represented as a twisted group C*-algebra of $\mathbf{Z}^n$, with respect to a bicharacter β, and written as $C^*(\mathbf{Z}^n, \beta)$. The two definitions may be identified as follows:

Let $e_1, \ldots, e_n$ be a basis of $\mathbf{Z}^n$, and denote by χ_{e_j} the characteristic function on $\mathbf{Z}^n$ supported at e_j; if we write $\beta(e_i, e_j)\overline{\beta(e_j, e_i)}$ in the form $\exp(-2\pi i\theta_{ij})$, we can identify χ_{e_j} with U_j as in $(*1)$, for $1 \leqq j \leqq n$.

We shall feel free to use both definitions and this identification.

For the importance of this class of C*-algebras, we refer to [R4, P and Po].

Built on [R3, EE, EL1 and EL2], we are now able to show that every simple 3-torus is an inductive limit of direct sums of four circle algebras. Consequently (applying Corollary 1 of the main theorem in [Po], also see [Pa] and the appendix of our paper for a direct proof), any non-commutative 3-torus is an inductive limit of type I C*-algebras.

Combining the classification theorem of Elliott (Theorem 7.1 of [E]), we see that the ordered K_0-group with a distinguished order unit is a complete isomorphic