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Supersymmetry, Vacuum Statistics, and the Fundamental Theorem of Algebra

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Abstract: I give an interpretation of the fundamental theorem of algebra based on supersymmetry and the Witten index. The argument gives a physical explanation of why a real polynomial of degree n need not have n real zeroes, while a complex polynomial of degree n must have n complex zeroes. This paper also addresses in a general and model-independent way the statistics of the perturbative ground states (the states which correspond to classical vacua) in supersymmetric theories with complex and with real superfields.

Supersymmetry provides some of the richest insights into the connections between physics and mathematics, with the Witten index [5] serving as one of the central tools in forging such connections. Perhaps what is most striking is the range of the applications of supersymmetry to mathematics; supersymmetry has been used to prove the Atiyah–Singer index theorem [2], to compute the topological invariants of manifolds [5, 6], and to derive a variety of results in arithmetic number theory [3]. The central role of the Witten index in these and in many other physical and mathematical applications stems from the invariance of the index under deformations of the parameters of a theory. This makes the index a powerful tool. It means that the index may be calculated reliably by simple means, as one need only find one point in parameter space where it is easily calculable to know its value at all points in parameter space; this in turn makes possible the derivation of exact, non-perturbative results about physical theories and the mathematical structures they describe (subject to certain caveats I mention below).

In this paper, using arguments that are not mathematically rigorous but which are nonetheless instructive and compelling, I extend the scope of the connections between supersymmetric physics and mathematical results by showing how one can use supersymmetry to obtain the fundamental theorem of algebra. In fact, I will use supersymmetry not only to show that an n^{th}-degree polynomial over the complex numbers always has n roots, but also to demonstrate that an n^{th}-degree polynomial over the reals has an even or odd number of real roots, according to whether