Commun. Math. Phys. 176, 681 – 700 (1996)

Communications in Mathematical Physics

Integrable Time-Discretisation of the Ruijsenaars–Schneider Model

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Received: 19 December 1994

Abstract: An exactly integrable symplectic correspondence is derived which in a continuum limit leads to the equations of motion of the relativistic generalization of the Calogero–Moser system, that was introduced for the first time by Ruijsenaars and Schneider. For the discrete-time model the equations of motion take the form of Bethe Ansatz equations for the inhomogeneous spin-$\frac{1}{2}$ XYZ Heisenberg magnet. We present a Lax pair, the symplectic structure and prove the involutivity of the invariants. Exact solutions are investigated in the rational and hyperbolic (trigonometric) limits of the system that is given in terms of elliptic functions. These solutions are connected with discrete soliton equations. The results obtained allow us to consider the Bethe Ansatz equations as ones giving an integrable symplectic correspondence mixing the parameters of the quantum integrable system and the parameters of the corresponding Bethe wavefunction.

1. Introduction

In some previous papers, [1, 2], cf. also [3], an exact time-discretization of the famous Calogero–Moser (CM) model, [4–7], was introduced and investigated. The discrete model is an integrable symplectic correspondence, (for a definition cf. [8]), that in a well-defined continuum limit yields the classical equations of motion of the CM system.[4] A few years ago Ruijsenaars and Schneider introduced in [10], cf. also [11, 12], a relativistic variant of the CM model, which is a parameter-deformation of the original model. The equations of motion of this system in its generic (elliptic) form read

$$\ddot{q}_i = \sum_{j \neq i} \dot{q}_i \dot{q}_j v(q_i - q_j), \quad i = 1, \ldots, N \ , \tag{1.1a}$$

★ Supported by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)
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[4] It should be noted that the discrete CM model can be inferred also from the Bäcklund transformations for the continuous CM model, that were presented in [9].