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Asymptotic Formulae for Lorenz and Horseshoe Knots

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Abstract: We derive various asymptotic formulae for the numbers of closed orbits in the Lorenz and Horseshoe templates with given knot invariants, (for example braid index and genus). We indicate how these estimates can be applied to more complicated flows by giving a bound for the genus of knotted periodic orbits in the 'figure of eight' template.

0. Introduction

Let Φ_t be a flow on S^3 (for example a Lorenz flow or an Axiom A no cycles flow) with countably many periodic orbits $(\tau_n)_{n=1}^{\infty}$. We regard each closed orbit as a knot in S^3. The set of all infinite collections of knots $(K_n)_{n=1}^{\infty}$ has the cardinality of the continuum. However, for Axiom A no cycle flows for example, the set of such collections of knots which occur as periodic orbits is countable, so only special ones can occur. The central problem in the study of knotted periodic orbits of flows in S^3 is to classify these families of knots, or more realistically to find restrictions on them.

In his survey lecture [W4], Williams suggested that a useful approach to this problem would be to associate a knot invariant $ki(\tau)$ to each closed orbit τ (e.g. braid index, genus) and to find restrictions on the sequences $(ki(\tau_n))_{n=1}^{\infty}$. In this paper, we provide a solution for three specific examples of flows. For various knot invariants, we show that such sequences must satisfy precise asymptotic formulae or bounds.

In [BW1] and [BW2], Birman and Williams introduced the notion of a template in S^3, which consists of a branched two manifold, with charts of two specific types, together with an expanding semiflow defined on it. For certain types of flows in S^3, one can construct a template for the flow in such a way that the periodic orbits of the flow and the semiflow correspond one-to-one, and this correspondence preserves knot types. (Strictly, one may first need to exclude finitely many orbits.) So we will study only knotted periodic orbits in templates.

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