

$\mathcal{W}_{1+\infty}$ and $\mathcal{W}(gl_N)$ with Central Charge N

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Abstract: We study representations of the central extension of the Lie algebra of differential operators on the circle, the $\mathcal{W}_{1+\infty}$ algebra. We obtain complete and specialized character formulas for a large class of representations, which we call primitive; these include all quasi-finite irreducible unitary representations. We show that any primitive representation with central charge N has a canonical structure of an irreducible representation of the \mathcal{W} -algebra $\mathcal{W}(gl_N)$ with the same central charge and that all irreducible representations of $\mathcal{W}(gl_N)$ with central charge N arise in this way. We also establish a duality between “integral” modules of $\mathcal{W}(gl_N)$ and finite-dimensional irreducible modules of gl_N , and conjecture their fusion rules.

Introduction

The Lie algebra $\widehat{\mathcal{D}}$, which is the unique non-trivial central extension of the Lie algebra \mathcal{D} of differential operators on the circle [KP1], has appeared recently in various models of two-dimensional quantum field theory and integrable systems, cf., e.g., [BK, FKN, PRS, IKS, CTZ, ASvM]. A systematic study of representation theory of the Lie algebra $\widehat{\mathcal{D}}$, which is often referred to as $\mathcal{W}_{1+\infty}$ algebra, was initiated in [KR]. In that paper irreducible quasi-finite highest weight representations of $\widehat{\mathcal{D}}$ were classified and it was shown that they can be realized in terms of irreducible highest weight representations of the Lie algebra of infinite matrices.

In the first part of the present paper we recall some of the results of [KR] and, as an immediate corollary, obtain complete and specialized character formulas for an arbitrary *primitive* representation of $\widehat{\mathcal{D}}$. (A primitive representation of $\widehat{\mathcal{D}}$ is an “analytic continuation” of a quasi-finite irreducible unitary representation of $\widehat{\mathcal{D}}$.) The results of [KR] were used previously in [Mat, AFOQ] to derive character formulas of primitive representations of central charge $c = 1$.

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