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Fusion Categories Arising from Semisimple Lie Algebras

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Abstract: Using tilting modules we equip certain semisimple categories with a "reduced" tensor product structure. The fusion rules for this tensor product are determined via known character formulas for the involved modules.

Introduction

In this paper the term "fusion rules" will cover the problem of describing the various decomposition multiplicities of the tensor structure on a given rigid braided tensor category.

Given a finite type Cartan datum one can associate at least 4 interesting categories to this. Namely

(1) The category $\mathscr{O}$ of the corresponding semisimple Lie algebra $\mathfrak{g}$.
(2) The category of rational modules of the corresponding semisimple, simply connected algebraic group G defined over a field of positive characteristic.
(3) The category of locally finite modules of the associated quantum algebra U specialized at an l^{th} root of unity.
(4) The category $\tilde{\mathscr{O}}_\kappa$ of fixed level representations of the affine Kac-Moody algebra $\tilde{\mathfrak{g}}$ associated to $\mathfrak{g}$.

In each of the cases (1–3) we shall investigate a certain semisimple subcategory equipped with a "reduced" tensor product and we shall prove some "fusion rules" in each case; these will be given in terms of the characters of the involved modules. Hence they will of course essentially be old character formulas in a new guise. Our approach will be in the framework of tilting modules.

The last case is treated in the thesis by Finkelberg [F] where he uses the "level-preserving" tensor product on the category $\tilde{\mathscr{O}}$ (see Kazhdan and Lusztig [KL2]) to relate the Grothendieck ring of $\tilde{\mathscr{O}}_\kappa$ with a quotient of $\mathscr{R}$ defined in 1.15. We state his precise result in 1.24.

The paper is divided into 3 parts dealing with the three above mentioned situations (1–3) respectively. Also in the first part we investigate the category $\mathscr{O}_S$ and in the