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Supersymmetry and Fredholm Modules Over Quantized Spaces

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Abstract: The purpose of this paper is to apply the framework of non-commutative differential geometry to quantum deformations of a class of Kähler manifolds. For the examples of the Cartan domains of type I and flat space, we construct Fredholm modules over the quantized manifolds using the supercharges which arise in the quantization of supersymmetric generalizations of the manifolds. We compute an explicit formula for the Chern character on generators of the Toeplitz $\mathbb{C}^*$-algebra.

I. Introduction

I.A. Since the early work on quantum mechanics ([15, 3, 12, 4]) by Heisenberg, Born, Jordan, and Dirac, it has been generally recognized that ordinary geometry does not apply to the subatomic world. In order to describe the physical phenomena in that world, the classical notion of phase space needs to be replaced by a non-commutative algebra of "quantum observables." The coordinates p and q on the phase space $\mathbb{R}^2$ are replaced by generators p and q that obey the famous commutation relation $[q, p] = i\hbar$. This "quantization" procedure amounts to studying a non-commutative deformation of a flat space, and, from a geometric viewpoint quantum mechanics emerges as some form of symplectic geometry on this non-commutative space. The classical algebra of functions on phase space arises as the $\hbar \to 0$ limit of the deformed algebra, and the Poisson bracket of two observables turns out to be the subleading term in the small $\hbar$ expansion of the commutator of the corresponding quantized observables. Much work has been done since the early quantum mechanics on extending this procedure to more general, non-flat phase spaces, resulting in powerful theories known as geometric quantization, deformation quantization, quantum groups, etc.

I.B. In the mid-eighties A. Connes [9] proposed a general scheme of non-commutative differential geometry which is ideally suited to describe the geometry of

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