Commun. Math. Phys. 161, 289–309 (1994)

Communications in Mathematical Physics

Level Spacing Distributions and the Bessel Kernel

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Received: 19 April 1993

Abstract: Scaling models of random $N \times N$ hermitian matrices and passing to the limit $N \to \infty$ leads to integral operators whose Fredholm determinants describe the statistics of the spacing of the eigenvalues of hermitian matrices of large order. For the Gaussian Unitary Ensemble, and for many others as well, the kernel one obtains by scaling in the "bulk" of the spectrum is the "sine kernel" $\dfrac{\sin \pi(x-y)}{\pi(x-y)}$. Rescaling the GUE at the "edge" of the spectrum leads to the kernel $\dfrac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}$, where Ai is the Airy function. In previous work we found several analogies between properties of this "Airy kernel" and known properties of the sine kernel: a system of partial differential equations associated with the logarithmic differential of the Fredholm determinant when the underlying domain is a union of intervals; a representation of the Fredholm determinant in terms of a Painlevé transcendent in the case of a single interval; and, also in this case, asymptotic expansions for these determinants and related quantities, achieved with the help of a differential operator which commutes with the integral operator. In this paper we show that there are completely analogous properties for a class of kernels which arise when one rescales the Laguerre or Jacobi ensembles at the edge of the spectrum, namely

$$\frac{J_\alpha(\sqrt{x})\sqrt{y}J_\alpha'(\sqrt{y}) - \sqrt{x}J_\alpha'(\sqrt{x})J_\alpha(\sqrt{y})}{2(x-y)},$$

where $J_\alpha(z)$ is the Bessel function of order α. In the cases $\alpha = \mp\frac{1}{2}$ these become, after a variable change, the kernels which arise when taking scaling limits in the bulk of the spectrum for the Gaussian orthogonal and symplectic ensembles. In particular, an asymptotic expansion we derive will generalize ones found by Dyson for the Fredholm determinants of these kernels.