Commun. Math Phys. 160, 217–238 (1994)

Communications in
Mathematical
Physics
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Rotational Symmetry of Solutions of Some Nonlinear Problems in Statistical Mechanics and in Geometry

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Received: 26 February 1993/in revised form: 21 June 1993

Abstract: The method of moving planes is used to establish a weak set of conditions under which the nonlinear equation $-\Delta u(x) = V(|x|)e^{u(x)}$, $x \in \mathbb{R}^2$ admits only rotationally symmetric solutions. Additional structural invariance properties of the equation then yield another set of conditions which are not originally covered by the moving plane technique. For instance, nonmonotonic V can be accommodated. Results for $-\Delta u(y) = V(y)e^{u(y)} - c$, with $y \in S^2$, are obtained as well. A nontrivial example of broken symmetry is also constructed. These equations arise in the context of extremization problems, but no extremization arguments are employed. This is of some interest in cases where the extremizing problem is neither manifestly convex nor monotone under symmetric decreasing rearrangements. The results answer in part some conjectures raised in the literature. Applications to logarithmically interacting particle systems and geometry are emphasized.

1. Introduction and Main Results

The present paper is concerned with spatial symmetry and symmetry breaking of solutions of certain nonlinear field equations in two-dimensional domains without boundary. The presence of (symmetric) boundaries is known to catalyze symmetry, as well as its breaking, in many circumstances. In the absence of boundaries, on the contrary, such features reflect intrinsic properties of the system under study. Various approaches to this type of problem exist in the literature, the applicability of which depends in part on the specific problem under consideration. We will be using here the method of moving planes [A, GNN1, GNN2, Li, LN1, LN2, LN3, CheL], which is based on the maximum principle and allows us to prove conditions under which *any* solution of our equations is symmetric.

We recall that so far, the moving plane method has shown the absence of symmetry breaking, under mild conditions, for positive solutions of the conformally invariant equations

$$\Delta u + u^p = 0; \quad x \in \mathbb{R}^n \tag{1.1a}$$