Commun. Math. Phys. 158, 45–66 (1993)

Communications in
Mathematical
Physics

Spectral Properties of One-Dimensional Schrödinger Operators with Potentials Generated by Substitutions

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Received: 21 May 1992/in revised form: 10 January 1993

Abstract: We investigate one-dimensional discrete Schrödinger operators whose potentials are invariant under a substitution rule. The spectral properties of these operators can be obtained from the analysis of a dynamical system, called the trace map. We give a careful derivation of these maps in the general case and exhibit some specific properties. Under an additional, easily verifiable hypothesis concerning the structure of the trace map we present an analysis of their dynamical properties that allows us to prove that the spectrum of the underlying Schrödinger operator is singular and supported on a set of zero Lebesgue measure. A condition allowing to exclude point spectrum is also given. The application of our theorems is explained on a series of examples.

1. Introduction

In this article we present general results on the spectral properties of a class of one-dimensional discrete Schrödinger operators of the form

$$H_v = -\Delta + V \quad \text{on } l^2(\mathbb{Z}), \tag{1.1}$$

where Δ is the discrete Laplacian and V is a diagonal operator whose diagonal elements V_n are obtained from a *substitution sequence* [1]. By a substitution sequence we mean the following. Let $\mathscr{A}$ be a finite set, called an *alphabet*. Let $\mathscr{A}^k$ be the set of *words* of length k in the alphabet, $\mathscr{A}^* \equiv \bigcup_{k\in\mathbb{N}} \mathscr{A}^k$ the set of all words of finite length, and $\mathscr{A}^{\mathbb{N}}$ the set of one-sided infinite sequences of letters. A map $\xi\colon \mathscr{A} \to \mathscr{A}^*$ is called a *substitution*. A substitution ξ naturally induces maps from $\mathscr{A}^* \to \mathscr{A}^*$ and $\mathscr{A}^{\mathbb{N}} \to \mathscr{A}^{\mathbb{N}}$, which we will denote by the same name and which are obtained simply be applying ξ to each letter in the respective words or sequences (e.g. $\xi(abc) = \xi(a)\xi(b)\xi(c)$). A substitution may possess fix-points in $\mathscr{A}^{\mathbb{N}}$, and such fix-points, u, will be called substitution sequences. There are two natural conditions that guarantee the existence of at least one fix-point, namely $\xi^\infty 0$, and that we will assume to be satisfied for all substitutions we discuss [1]: