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# Generalized Drinfel'd-Sokolov Hierarchies

## II. The Hamiltonian Structures

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**Abstract.** In this paper we examine the bi-Hamiltonian structure of the generalized KdV-hierarchies. We verify that both Hamiltonian structures take the form of Kirillov brackets on the Kac-Moody algebra, and that they define a coordinated system. Classical extended conformal algebras are obtained from the second Poisson bracket. In particular, we construct the $W_n^{(l)}$ algebras, first discussed for the case $n = 3$ and $l = 2$ by Polyakov and Bershadsky.

## 1. Introduction

This paper is a continuation of [1], where we generalized the Drinfel'd-Sokolov construction of integrable hierarchies of partial differential equations from Kac-Moody algebras, see [2]. The work of Drinfel'd and Sokolov, itself, constituted a generalization of the original Korteweg-de Vries (KdV) hierarchy, the archetypical integrable system. The main omission from our previous paper was a discussion of the Hamiltonian formalism of these integrable hierarchies, which is the subject of this present paper. A Hamiltonian analysis of these integrable systems allows a much deeper insight into their structure, in particular important algebraic structures are encountered such as the Gel'fand-Dikii algebras [3], or classical $W$-algebras, which arise as the second Hamiltonian structure of the $A_n$-hierarchies of Drinfel'd and Sokolov. Though we shall say more about this later, the exemplar of this connexion is found in the original KdV hierarchy whose second Hamiltonian structure is the Virasoro algebra. The new hierarchies of [1] lead amongst other things to the $W_N^{(l)}$-algebras for $1 \leq l \leq N-1$, introduced in [4].

A feature often encountered in the Hamiltonian analysis of integrable hierarchies, is the presence of two *coordinated* Poisson structures which we designate $\{\phi, \psi\}_1$ and $\{\phi, \psi\}_2$. The property of *coordination* implies that the one-parameter family of brackets

$$\{\phi, \psi\} = \{\phi, \psi\}_1 + \mu\{\phi, \psi\}_2 ,$$